Solutions to Tutorial Sheet 7

1. Let $F(x, y, z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of F at $(1, -1, 3)$ and the equations of the tangent plane and the normal line to the surface $F(x, y, z) = 7$ at $(1, -1, 3)$.

   Solution. $(\nabla F)(1, -1, 3) = \left(\frac{\partial F}{\partial x}(1, -1, 3), \frac{\partial F}{\partial y}(1, -1, 3), \frac{\partial F}{\partial z}(1, -1, 3)\right) = 4\mathbf{j} + 6\mathbf{k}$.

   The tangent plane to the surface $F(x, y, z) = 7$ at the point $(1, -1, 3)$ is given by,

   $$0 \times (x - 1) + 4 \times (y + 1) + 6 \times (z - 3) = 0 \implies 2y + 3z = 7.$$

   The normal line to the surface $F(x, y, z) = 7$ at the point $(1, -1, 3)$ is given by $x = 1, 3y - 2z + 9 = 0$.

2. Find $D_{\vec{u}}F(2, 2, 1)$, where $F(x, y, z) = 3x - 5y + 2z$, and $\vec{u}$ is the unit vector in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2, 2, 1)$.

   Solution. $\vec{u} = \frac{(2, 2, 1)}{\sqrt{2^2 + 2^2 + 1^2}} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = \frac{2}{3}(\mathbf{i} + \mathbf{j})\frac{1}{3}\mathbf{k}$ and $(\nabla F)(2, 2, 1) = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$. Therefore, $D_{\vec{u}}F(2, 2, 1) = (\nabla F)(2, 2, 1) \cdot \vec{u} = \frac{6}{3} - \frac{10}{3} + \frac{2}{3} = -\frac{2}{3}$.

3. Given $\sin(x + y) + \sin(y + z) = 1$, find $\frac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y + z) \neq 0$.

   Solution. Given that $\sin(x + y) + \sin(y + z) = 1$ (with $\cos(y + z) \neq 0$).

   You may assume that z is a sufficiently smooth function of x and y.

   Differentiating w.r.t. x while keeping y fixed, we get,

   $$\cos(x + y) + \cos(y + z)\frac{\partial z}{\partial x} = 0. \qquad (*)$$

   Similarly, differentiating w.r.t. y while keeping x fixed, we get,

   $$\cos(x + y) + \cos(y + z)\left(1 + \frac{\partial z}{\partial y}\right) = 0. \qquad (**)$$

   Differentiating $(*)$ w.r.t y we have,

   $$-\sin(x + y) - \sin(y + z)\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} + \cos(y + z)\frac{\partial^2 z}{\partial x \partial y} = 0.$$

   Thus, using $(*)$ and $(**)$, we have,

   $$\begin{aligned}\frac{\partial^2 z}{\partial x \partial y} &= \frac{1}{\cos(y + z)}\left[\sin(x + y) + \sin(y + z)\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}\right] \\ &= \frac{1}{\cos(y + z)}\left[\sin(x + y) + \sin(y + z)\left(-\frac{\cos(x + y)}{\cos(y + z)}\right)\left(-\frac{\cos(x + y)}{\cos(y + z)}\right)\right] \\ &= \frac{\sin(x + y)}{\cos(y + z)} + \tan(y + z)\left(\frac{\cos^2(x + y)}{\cos^2(y + z)}\right)\end{aligned}$$

4. If $f(0,0) = 0$ and

$$f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2} \text{ for } (x,y) \neq (0,0),$$

show that both f_{xy} and f_{yx} exist at $(0,0)$, but they are not equal. Are f_{xy} and f_{yx} continuous at $(0,0)$?

Solution. We have,

$$f_{xy}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k},$$

where (noting that $k \neq 0$),

$$f_x(0,k) = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} = -k \text{ and } f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

Therefore,

$$f_{xy}(0,0) = \lim_{k \to 0} \frac{-k - 0}{k} = -1 \text{ ; similarly } f_{yx}(0,0) = 1.$$

Thus,

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

By directly computing f_{xy}, f_{yx} for $(x,y) \neq (0,0)$, one observes that these are not continuous at $(0,0)$.

5. Show that the following functions have local minima at the indicated points:

(i) $f(x,y) = x^4 + y^4 + 4x - 32y - 7$, $(x_0, y_0) = (-1, 2)$

Solution. $f_x(-1,2) = 0 = f_y(-1,2)$, $H_f(-1,2) = \begin{bmatrix} 12 & 0 \\ 0 & 48 \end{bmatrix}$

$D(-1,2) = 12 \times 48 - 0^2 > 0$, $f_{xx}(-1,2) = 12 > 0 \implies (-1,2)$ is a point of local minimum of f.

(ii) $f(x,y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3$, $(x_0, y_0) = (0,0)$

Solution. $f_x(0,0) = 0 = f_y(0,0)$, $H_f(0,0) = \begin{bmatrix} 6 & -2 \\ -2 & 10 \end{bmatrix}$

$D(0,0) = 6 \times 10 - (-2)^2 > 0$, $f_{xx}(0,0) = 6 > 0 \implies (0,0)$ is a point of local minimum of f.

6. Analyze the following functions for local maxima, local minima and saddle points:

(i) $f(x,y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$

Solution. $f_x = e^{-(x^2+y^2)/2}(2x - x^3 + xy^2)$, $f_y = e^{-(x^2+y^2)/2}(-2y + y^3 - x^2 y)$.

Critical points are $(0,0)$, $(\pm\sqrt{2}, 0)$, $(0, \pm\sqrt{2})$.

$H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \implies (0,0)$ is a saddle point of f.

$H_f(\pm\sqrt{2}, 0) = \begin{bmatrix} -4/e & 0 \\ 0 & -4/e \end{bmatrix} \implies (\pm\sqrt{2}, 0)$ are points of local maximum of f.

$H_f(0, \pm\sqrt{2}) = \begin{bmatrix} 4/e & 0 \\ 0 & 4/e \end{bmatrix} \implies (0, \pm\sqrt{2})$ are points of local minimum of f.

(ii) $f(x, y) = x^3 - 3xy^2$

Solution. $f_x = 3x^2 - 3y^2$ and $f_y = -6xy$ imply that $(0, 0)$ is the only critical point of f. Now,

$$H_f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus the standard derivative test fails. However, $f(\pm\epsilon, 0) = \pm\epsilon^3$ for any ϵ so that $(0, 0)$ is a saddle point of f.

7. Find the absolute maximum and the absolute minimum of

$$f(x, y) = (x^2 - 4x)\cos y \text{ for } 1 \le x \le 3, -\pi/4 \le y \le \pi/4.$$

Solution. From $f(x, y) = (x^2 - 4x)\cos y$ $(1 \le x \le 3, -\pi/4 \le y \le \pi/4)$, we have,

$$f_x = (2x - 4)\cos y \text{ and } f_y = -(x^2 - 4x)\sin y.$$

Thus the only critical pointof f is $P = (2, 0)$. Note that $f(P) = -4$. Next, $g_\pm(x) \equiv f(x, \pm\frac{\pi}{4}) = \frac{(x^2 - 4x)}{\sqrt{2}}$ for $1 \le x \le 3$ has $x = 2$ as the only critical point so that we consider $P_\pm = (2, \pm\frac{\pi}{4})$. Note that $f(P_\pm) = \frac{-4}{\sqrt{2}}$. We also need to check $g_\pm(1) = f(1, \pm\frac{\pi}{4})$ $(\equiv f(Q_\pm))$ and $g_\pm(3) = f(3, \pm\frac{\pi}{4})$ $(\equiv f(S_\pm))$. Note that $f(Q_\pm) = \frac{-3}{\sqrt{2}}$ and $f(S_\pm) = -\frac{-3}{\sqrt{2}}$.

Next, consider $h(y) \equiv f(1, y) = -3\cos y$ for $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$. The only critical point of h is $y = 0$. Note that $h(0) = f(1, 0)$ $(\equiv f(M)) = -3$. ($h(\pm\frac{\pi}{4})$ is just $f(Q_\pm)$).

Finally, consider $k(y) = f(3, y) = -3\cos y$ for $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$. The only critical point of k is $y = 0$. Note that $k(0) = f(3, 0)$ $(\equiv f(T)) = -3$. ($k(\pm\frac{\pi}{4})$ is just $f(S_\pm)$).

Summarizing, we have the following table:

Points	P_+	P_-	Q_+	Q_-	S_+	S_-	T	P	M
Values	$-\frac{4}{\sqrt{2}}$	$-\frac{4}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3}{\sqrt{2}}$	-3	-4	-3

By inspection one finds that $f_{min} = -4$ attained at $P = (2, 0)$ and $f_{max} = \frac{-3}{\sqrt{2}}$ at $Q_\pm = (1, \pm\frac{\pi}{4})$ and at $S_\pm = (3, \pm\frac{\pi}{4})$.

Credits: MA105 Tutorial Solutions (2014)