Solutions to Tutorial Sheet 6

2. Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$:

   (i) $f(x, y) = x - y$

   Solution. A level curve corresponding to any of the given values of c is the straight line $x - y = c$ in the xy-plane. A contour line corresponding to any of the given values of c is the same line shifted to the plane $z = c$ in $\mathbb{R}^3$.

   (ii) $f(x, y) = x^2 + y^2$

   Solution. Level curves do not exist for $c = -3, -2, -1$. The level curve corresponding to $c = 0$ is the point $(0, 0)$. The level curves corresponding to $c = 1, 2, 3, 4$ are concentric circles centered at the origin in the xy-plane. Contour lines corresponding to $c = 1, 2, 3, 4$ are the cross-sections in $\mathbb{R}^3$ of the paraboloid $z = x^2 + y^2$ by the plane $z = c$, i.e., circles in the plane $z = c$ centered at $(0, 0, c)$.

   (iii) $f(x, y) = xy$

   Solution. For $c = -3, -2, -1$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the second and fourth quadrant. For $c = 1, 2, 3, 4$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the first and third quadrant. For $c = 0$, the corresponding level curve (resp. the contour line) is the union of the x-axis and the y-axis in the xy-plane (resp. in the xyz-space). A contour line corresponding to a non-zero c is the cross-section of the hyperboloid $z = xy$ by the plane $z = c$, i.e., a rectangular hyperbola in the plane $z = c$.

3. Using definition, examine the following functions for continuity at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero:

   (i) $\dfrac{x^3 y}{x^6 + y^2}$

   Solution. Discontinuous at $(0, 0)$ (check $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ using $y = mx^3$).

   (ii) $xy\dfrac{x^2 - y^2}{x^2 + y^2}$

   Solution. Continuous at $(0, 0)$ (since $\left| xy\dfrac{x^2 - y^2}{x^2 + y^2} \right| \leq |xy|$).

   (iii) $||x| - |y|| - |x| - |y|$

   Solution. Continuous at $(0, 0)$ (since $|f(x, y)| \leq 2(|x| + |y|) \leq 4\sqrt{x^2 + y^2}$).

6. Examine the following functions for the existence of partial derivatives at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero:

   (i) $xy\dfrac{x^2 - y^2}{x^2 + y^2}$

Solution. $f_x(x,y) = y\left(1 - \frac{2y^4}{(x^2+y^2)^2}\right)$ and $f_y(x,y) = x\left(\frac{2x^4}{(x^2+y^2)^2} - 1\right)$, hence $|f_x(x,y)| \leq |y|$ and $|f_y(x,y)| \leq |x|$ (since $0 \leq \frac{y^4}{(x^2+y^2)^2} \leq 1$ and $0 \leq \frac{x^4}{(x^2+y^2)^2} \leq 1$). So $f_x(0,0) = 0 = f_y(0,0)$.

(ii) $\frac{\sin^2(x+y)}{|x|+|y|}$

Solution. $f_x(0,0) = \lim_{h\to 0} \frac{\sin^2(h)/|h|}{h} = \lim_{h\to 0} \frac{\sin^2(h)}{h|h|}$ does not exist (because LHL $\neq$ RHL).

Similarly, $f_y(0,0)$ does not exist.

7. Let $f(0,0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \text{ for } (x,y) \neq (0,0).$$

Show that f is continuous at $(0,0)$, and the partial derivatives of f exist but are not bounded in any disc (howsoever small) around $(0,0)$.

Solution. $|f(x,y)| \leq x^2+y^2 \implies f$ is continuous at $(0,0)$. Now,

$$f_x = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right)$$

It is easily checked that $f_x(0,0) = f_y(0,0) = 0$. The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at $(0,0)$, while $\frac{2x}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)$ is unbounded in any such disc (consider for example $(x,y) = \left(\frac{1}{\sqrt{n\pi}}, 0\right)$ for n a large positive integer). Thus $f_x(0,0)$ is unbounded in any disc around $(0,0)$.

8. Let $f(0,0) = 0$ and

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y) & x \neq 0, y \neq 0 \\ x\sin(1/x) & x \neq 0, y = 0 \\ y\sin(1/y) & y \neq 0, x = 0 \end{cases}$$

Show that none of the partial derivatives of f exist at $(0,0)$ although f is continuous at $(0,0)$.

Solution. $f_x(0,0) = \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\sin\frac{1}{h}$ does not exist. Similarly $f_y(0,0)$ does not exist. Clearly, f is continuous at $(0,0)$.

9. Examine the following functions for the existence of directional derivatives and differentiability at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

(i) $xy\frac{x^2-y^2}{x^2+y^2}$

Solution. Let $\vec{v} = (a,b)$ be any unit vector in $\mathbb{R}^2$. We have

$$(D_{\vec{v}}f)(0,0) = \lim_{h\to 0}\frac{f(h\vec{v})}{h} = \lim_{h\to 0}\frac{f(ha,hb)}{h} = \lim_{h\to 0}\frac{h^2ab\left(\frac{a^2-b^2}{a^2+b^2}\right)}{h} = 0.$$

Therefore $(D_{\vec{v}}f)(0,0)$ exists and equals 0 for every unit vector $\vec{v}$ in $\mathbb{R}^2$. For considering differentiability, note that $f_x(0,0) = (D_{\hat{i}}f)(0,0) = 0 = f_y(0,0) = (D_{\hat{j}}f)(0,0)$. We have then

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk(h^2-k^2)|}{(h^2+k^2)^{3/2}} = 0,$$

since

$$0 \leq \frac{|hk(h^2-k^2)}{(h^2+k^2)^{3/2}} \leq \frac{|hk|}{\sqrt{h^2+k^2}} \leq \sqrt{h^2+k^2}.$$

Thus f is differentiable at $(0,0)$.

(ii) $\dfrac{x^3}{x^2+y^2}$

Solution. Note that for any unit vector $\vec{v} = (a,b)$ in $\mathbb{R}^2$, we have

$$(D_{\vec{v}}f)(0,0) = \lim_{h\to 0} \frac{h^3a^3}{h(h^2(a^2+b^2))} = \lim_{h\to 0} \frac{a^3}{(a^2+b^2)} = \frac{a^3}{(a^2+b^2)}.$$

To consider differentiability, note that $f_x(0,0) = 1$, $f_y(0,0) = 0$ and

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - h\times 1 - k\times 0|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|h^3/(h^2+k^2) - h|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk^2|}{(h^2+k^2)^{3/2}}$$

does not exist (consider $k = mh$, and then put $m = 10^{-5}$). Hence f is not differentiable at $(0,0)$.

(iii) $(x^2+y^2)\sin\dfrac{1}{x^2+y^2}$

Solution. For any unit vector $\vec{v} \in \mathbb{R}^2$, one has,

$$(D_{\vec{v}}f)(0,0) = \lim_{h\to 0} \frac{h^2(a^2+b^2)\sin\left[\frac{1}{h^2(a^2+b^2)}\right]}{h} = 0.$$

Also,

$$\lim_{(h,k)\to(0,0)} \frac{\left|(h^2+k^2)\sin\left[\frac{1}{h^2+k^2}\right]\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right) = 0.$$

Therefore f is differentiable at $(0,0)$.

10. Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2} \text{ if } y \neq 0.$$

Show that f is continuous at $(0,0)$, $(D_{\vec{v}}f)(0,0)$ exists for every vector $\vec{v}$, yet f is not differentiable at $(0,0)$.

Solution. $f(0,0) = 0$, $|f(x,y)| \leq \sqrt{x^2+y^2} \implies f$ is continuous at $(0,0)$. Let $\vec{v}$ be a unit vector in $\mathbb{R}^2$. For $\vec{v} = (a,b)$, with $b \neq 0$, one has,

$$(D_{\vec{v}}f)(0,0) = \lim_{h\to 0} \frac{1}{h}\frac{hb}{|hb|}\sqrt{h^2a^2+h^2b^2} = \frac{(\sqrt{a^2+b^2})b}{|b|}.$$

If $\vec{v} = (a, 0)$, then $(D_{\vec{v}}f)(0,0) = 0$. Hence $(D_{\vec{v}}f)(0,0)$ exists for every unit vector $\vec{v} \in \mathbb{R}^2$. Further,

$$f_x(0,0) = 0,\ f_y(0,0) = 1,$$

and

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - 0 - h \times 0 - k \times 1|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\left|\frac{k}{|k|}\sqrt{h^2+k^2} - k\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \left|\frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}}\right|$$

does not exist (consider $h = mk$ and then put $m = 10^{-5}$) so that f is not differentiable at $(0,0)$.

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