Solutions to Tutorial Sheet 3

1. Show that the cubic $x^3 - 6x + 3$ has all roots real.

   Solution. $f(x) = x^3 - 6x + 3$ has stationary points at $x = \pm\sqrt{2}$. Note that $f(-\sqrt{2}) = 4\sqrt{2} + 3 > 0$ and $f(\sqrt{2}) = 3 - 4\sqrt{2} < 0$. Therefore, f has a root in $(-\sqrt{2}, \sqrt{2})$. Also, $\lim_{x\to-\infty} f(x) \to -\infty$ and so f has a root in $(-\infty, -\sqrt{2})$ and $\lim_{x\to\infty} f(x) \to \infty$ implies f has a root in $(\sqrt{2}, \infty)$. Since f has at host 3 roots, all the roots are real.

4. Consider the cubic $f(x) = x^3 + px + q$, where p and q are real numbers. If f has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:

   (i) $p < 0$.

   Solution. Since f has 3 distinct roots, say $r_1 < r_2 < r_3$, by Rolle's theorem, $f'(x)$ has at least 2 real roots, say x_1 and x_2 such that $r_1 < x_1 < r_2$ and $r_2 < x_2 < r_3$. Since $f'(x) = 3x^2 + p$, this implies that $p < 0$, so that 2 real roots exist.

   (ii) f has maximum/minimum at $\pm\sqrt{-p/3}$.

   Solution. Solving $f'(x) = 0$, we get $x_1 = -\sqrt{-p/3}$ and $x_2 = \sqrt{-p/3}$. Now $f''(x_1) = 6x_1 < 0$ and so f has a local maxima at $x = x_1$. Similarly, f has a local minima at $x = x_2$.

   (iii) The maximum/minimum values are of opposite signs.

   Solution. Since $f'(x)$ is negative between its roots x_1 and x_2 and f has a root r_2 in (x_1, x_2), we must have $f(x_1) > 0$ and $f(x_2) < 0$. Further,

   $$f(x_1) = q + \sqrt{\frac{-4p^3}{27}}, f(x_2) = q - \sqrt{\frac{-4p^3}{27}}$$

   so that

   $$\frac{4p^3 + 27q^2}{27} = f(x_1)f(x_2) < 0.$$

5. Use the MVT to prove $|\sin a - \sin b| \leq |a - b$ for all $a, b \in \mathbb{R}$.

   Solution. For some c between a and b, one has, by MVT,

   $$\left|\frac{\sin a - \sin b}{a - b}\right| = |\cos c| \leq 1$$

7. Let $a > 0$ and f be continuous on $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.

   Optional. Show that under the given conditions, in fact $f(x) = x$ for every x.

   Solution. By Lagrange's MVT, there exists $c_1 \in (-a, 0)$ and there exists $c_2 \in (0, a)$ such that

   $$f(0) - f(-a) = a \times f'(c_1) \text{ and } f(a) - f(0) = a \times f'(c_2)$$

Using the given conditions, we obtain

$$f(0) + a \leq a \text{ and } a - f(0) \leq a$$

which implies $f(0) = 0$.

Hint. For the optional part, consider $g(x) = f(x) - x$, $x \in [-a, a]$.

Solution. Since $g'(x) = f'(x) - 1 \leq 0$, g is decreasing over $[-a, a]$. As $g(-a) = g(a) = 0$, we have $g \equiv 0$.

8. In each case, find a function f which satisfies all the given conditions, or else show that no such function exists.

(i) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 1$

Solution. No such function exists in view of Rolle's theorem on $[0, 1]$.

(ii) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$.

Solution. $f(x) = x + \frac{x^2}{2}$

(iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$

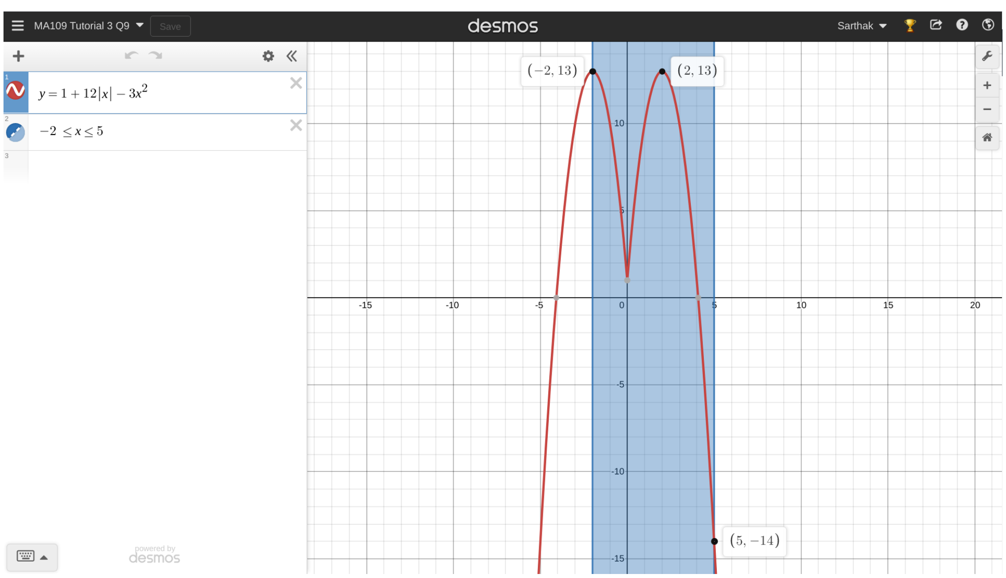
Solution. $f''(x) \geq 0 \implies f'$ increasing. As $f'(0) = 1$, by Lagrange's MVT we have $f(x) - f(0) \geq x$ for $x > 0$. Hence f with the required properties cannot exist.

(iv) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 1$ for all $x < 0$

Solution.

$$f(x) = \begin{cases} \frac{1}{1-x} & x \leq 0 \\ 1 + x + x^2 & x > 0 \end{cases}$$

9. Let $f(x) = 1 + 12|x| - 3x^2$. Find the absolute maximum and the absolute minimum of f on [-2,5]. Verify it from the sketch of the curve $y = f(x)$ on [-2,5].



Solution. The points to check are the end points $x = -2$ and $x = 5$, the point of non-differentiability $x = 0$ and the stationary point $x = 2$. The values of f at these points are given by

$$f(-2) = f(2) = 13, f(0) = 1, f(5) = -14.$$

Thus, global maximum is 13 at $x = \pm 2$ and global minimum is -14 at $x = 5$.

10. A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass and the semicircular portion is to be of colored glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window frame is to be p feet, find the dimensions of the window which will admit the maximum light.

Solution. Let the dimensions of the rectangular portion be l and b. Then the radius of the semicircular portion will be $r = b/2$. Then total perimeter is given by

$$p = 2l + b + \pi r = 2l + \left(1 + \frac{\pi}{2}\right) b.$$

Assume that clear glass allows 1 unit of light per square foot. Then the total light admitted by the window is given as

$$L = l \times b \times 1 + \frac{\pi r^2}{2} \times \frac{1}{2} = lb + \frac{\pi}{16} b^2$$

Using the previous relation, we can eliminate l, and get

$$L = \frac{\pi}{16} b^2 + b \left(\frac{p - \left(1 + \frac{\pi}{2}\right) b}{2} \right) = b^2 \left(-\frac{3\pi + 8}{16} \right) + b \left(\frac{p}{2} \right)$$

To maximise L with respect to b, we set $\frac{dL}{db}$ to 0. This gives us

$$2b \left(-\frac{3\pi + 8}{16} \right) + \frac{p}{2} = 0$$

Hence the dimensions to maximise admitted light are

$$b = \frac{4p}{3\pi + 8} \text{ and } l = \frac{(\pi + 4)p}{2(3\pi + 8)}$$

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