Solutions to Tutorial Sheet 1

1. Using $(\epsilon - n_0)$ definition, prove the following:

(iii) $\lim\limits_{n\to\infty} \frac{n^{2/3} sin(n!)}{n+1} = 0$

Solution. For a given $\epsilon > 0$, we have to find $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n \geq n_0$. Note that

$$|a_n| < \frac{n^{2/3}}{n+1} < \frac{1}{n^{1/3}}$$

Hence, select $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\epsilon^3}$. *(Think about why is this always possible.)*

(iv) $\lim\limits_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right) = 0$

Solution. Following approach similar to previous part, note that

$$|a_n| = \frac{1}{n}\left(2 - \frac{1}{n+1}\right) < \frac{2}{n}$$

Hence, select $n_0 \in \mathbb{N}$ such that $n_0 > \frac{2}{\epsilon}$. *(Think again. Same logic.)*

2. Show that the following limits exist and find them:

(i) $\lim\limits_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$

Solution.

$$\frac{n^2}{n^2+n} \leq a_n \leq \frac{n^2}{n^2+1} \implies \lim_{n\to\infty} a_n = 1$$

(iv) $\lim\limits_{n\to\infty} (n)^{1/n}$

Solution. Let $n^{1/n} = 1 + h_n$. For $n \geq 2$, we have

$$n = (1+h_n)^n \geq 1 + nh_n + \binom{n}{2}h_n^2 > \binom{n}{2}h_n^2 \implies 0 < h_n^2 < \frac{2}{n-1} \implies \lim_{n\to\infty} h_n = 0 \implies \lim_{n\to\infty} a_n = 1$$

(v) $\lim\limits_{n\to\infty} \left(\frac{cos\pi\sqrt{n}}{n^2}\right)$

Solution.

$$0 < \left|\frac{cos\pi\sqrt{n}}{n^2}\right| \leq \frac{1}{n^2} \implies \lim_{n\to\infty} a_n = 0$$

(vi) $\lim\limits_{n\to\infty} (\sqrt{n}(\sqrt{n+1} - \sqrt{n}))$

Solution.

$$\sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{1 + \sqrt{1 + \frac{1}{n}}} \implies \lim_{n\to\infty} a_n = \frac{1}{2}$$

3. Show that the following sequences are not convergent:

(i) $\{\frac{n^2}{n+1}\}_{n\geq 1}$

Solution.

$$\frac{n^2}{n+1} = (n-1) + \frac{1}{n+1} \text{ is not convergent since } \lim_{n\to\infty} \frac{1}{n+1} = 0$$

4. Determine whether the sequences are increasing or decreasing:

   (i) $\{\frac{n}{n^2+1}\}_{n\geq 1}$

   Solution. Decreasing, since $a_n = \frac{1}{n+\frac{1}{n}}$ and $\{n+\frac{1}{n}\}_{n\geq 1}$ is increasing.

   (iii) $\{\frac{1-n}{n^2}\}_{n\geq 2}$

   Solution. Increasing, as $a_{n+1} - a_n = \frac{n(n-1)-1}{n^2(n+1)^2} > 0$ for $n \geq 2$.

5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits:

   (ii) $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2+a_n}$ $\forall n \geq 1$

   Solution. By induction, $\sqrt{2} \leq a_n < 2$ $\forall n$. Hence, $a_{n+1} - a_n = \frac{(2-a_n)(1+a_n)}{a_n+\sqrt{2+a_n}} > 0$ $\forall n$. Thus $\{a_n\}_{n\geq 2}$ is monotonically increasing and bounded above by 2. So $\lim\limits_{n\to\infty} a_n = a$ (say) exists, and $\sqrt{2} \leq a < 2$. Also, $a = \sqrt{2+a}$, id est, $a^2 = a+2 \implies a = -1,\ 2$. Hence $\lim\limits_{n\to\infty} a_n = 2$.

7. If $\lim\limits_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \text{ for all } n \geq n_0.$$

   Solution. Take $\epsilon = |L|/2$. Then $\epsilon > 0$ and since $a_n \to L$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ $\forall n \geq n_0$. Now $||a_n| - |L|| \leq |a_n - L|$ and hence $|a_n| > |L| - \epsilon = |L|/2$ $\forall n \geq n_0$.

8. If $a_n \geq 0$ and $\lim\limits_{n\to\infty} a_n = 0$, show that $\lim\limits_{n\to\infty} a_n^{1/2} = 0$.

   Optional: State and prove a corresponding result if $a_n \to L > 0$.

   Solution. Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon^2$ $\forall n \geq n_0$. Hence $|\sqrt{a_n}| < \epsilon$ $\forall n \geq n_0$.

   Hint. For optional part, try using the fact that a_n will be bounded and $a_n - L = (\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})$.

10. Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent if and only if both the sub-sequences $\{a_2 n\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

    Solution. The implication " $\implies$ " is obvious. For the converse, suppose both $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ converge to ℓ. Let $\epsilon > 0$ be given. Choose $n_1,\ n_2 \in \mathbb{N}$ such that $|a_{2n} - \ell| < \epsilon$ for all $n \geq n_1$ and $|a_{2n+1} - \ell| < \epsilon$ for all $n \geq n_2$. Let $n_0 = max\{n_1, n_2\}$. Then $|a_n - \ell| < \epsilon$ for all $n \geq 2n_0 + 1$.

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