# Solutions to Tutorial Sheet 1

1. Using $(\epsilon - n_0)$ definition, prove the following:

   (iii) $\lim_{n\to\infty} \frac{n^{2/3} sin(n!)}{n+1} = 0$

   **Solution.** For a given $\epsilon > 0$, we have to find $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon$ for all $n \geq n_0$. Note that

   $$|a_n| < \frac{n^{2/3}}{n+1} < \frac{1}{n^{1/3}}$$

   Hence, select $n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\epsilon^3}$. *(Think about why is this always possible.)*

   (iv) $\lim_{n\to\infty} \left(\frac{n}{n+1} - \frac{n+1}{n}\right) = 0$

   **Solution.** Following approach similar to previous part, note that

   $$|a_n| = \frac{1}{n}\left(2 - \frac{1}{n+1}\right) < \frac{2}{n}$$

   Hence, select $n_0 \in \mathbb{N}$ such that $n_0 > \frac{2}{\epsilon}$. *(Think again. Same logic.)*

2. Show that the following limits exist and find them:

   (i) $\lim_{n\to\infty} \left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \cdots + \frac{n}{n^2+n}\right)$

   **Solution.**

   $$\frac{n^2}{n^2+n} \leq a_n \leq \frac{n^2}{n^2+1} \implies \lim_{n\to\infty} a_n = 1$$

   (iv) $\lim_{n\to\infty} (n)^{1/n}$

   **Solution.** Let $n^{1/n} = 1 + h_n$. For $n \geq 2$, we have

   $$n = (1+h_n)^n \geq 1 + nh_n + \binom{n}{2} h_n^2 > \binom{n}{2} h_n^2 \implies 0 < h_n^2 < \frac{2}{n-1} \implies \lim_{n\to\infty} h_n = 0 \implies \lim_{n\to\infty} a_n = 1$$

   (v) $\lim_{n\to\infty} \left(\frac{cos\pi\sqrt{n}}{n^2}\right)$

   **Solution.**

   $$0 < \left|\frac{cos\pi\sqrt{n}}{n^2}\right| \leq \frac{1}{n^2} \implies \lim_{n\to\infty} a_n = 0$$

   (vi) $\lim_{n\to\infty} (\sqrt{n}(\sqrt{n+1} - \sqrt{n}))$

   **Solution.**

   $$\sqrt{n}(\sqrt{n+1} - \sqrt{n}) = \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{1 + \sqrt{1 + \frac{1}{n}}} \implies \lim_{n\to\infty} a_n = \frac{1}{2}$$

3. Show that the following sequences are not convergent:

   (i) $\{\frac{n^2}{n+1}\}_{n\geq 1}$

   **Solution.**

   $$\frac{n^2}{n+1} = (n-1) + \frac{1}{n+1} \text{ is not convergent since } \lim_{n\to\infty} \frac{1}{n+1} = 0$$

4. Determine whether the sequences are increasing or decreasing:

(i) $\{\frac{n}{n^2+1}\}_{n\geq 1}$

**Solution.** Decreasing, since $a_n = \frac{1}{n+\frac{1}{n}}$ and $\{n + \frac{1}{n}\}_{n\geq 1}$ is increasing.

(iii) $\{\frac{1-n}{n^2}\}_{n\geq 2}$

**Solution.** Increasing, as $a_{n+1} - a_n = \frac{n(n-1)-1}{n^2(n+1)^2} > 0$ for $n \geq 2$.

5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits:

(ii) $a_1 = \sqrt{2}$, $a_{n+1} = \sqrt{2 + a_n}$ $\forall n \geq 1$

**Solution.** By induction, $\sqrt{2} \leq a_n < 2$ $\forall n$. Hence, $a_{n+1} - a_n = \frac{(2-a_n)(1+a_n)}{a_n+\sqrt{2+a_n}} > 0$ $\forall n$. Thus $\{a_n\}_{n\geq 2}$ is monotonically increasing and bounded above by 2. So $\lim\limits_{n\to\infty} a_n = a$ (say) exists, and $\sqrt{2} \leq a < 2$. Also, $a = \sqrt{2 + a}$, id est, $a^2 = a + 2 \implies a = -1,\ 2$. Hence $\lim\limits_{n\to\infty} a_n = 2$.

7. If $\lim\limits_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2} \text{ for all } n \geq n_0.$$

**Solution.** Take $\epsilon = |L|/2$. Then $\epsilon > 0$ and since $a_n \to L$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - L| < \epsilon$ $\forall n \geq n_0$. Now $||a_n| - |L|| \leq |a_n - L|$ and hence $|a_n| > |L| - \epsilon = |L|/2$ $\forall n \geq n_0$.

8. If $a_n \geq 0$ and $\lim\limits_{n\to\infty} a_n = 0$, show that $\lim\limits_{n\to\infty} a_n^{1/2} = 0$.

**Optional:** State and prove a corresponding result if $a_n \to L > 0$.

**Solution.** Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n| < \epsilon^2$ $\forall n \geq n_0$. Hence $|\sqrt{a_n}| < \epsilon$ $\forall n \geq n_0$.

*Hint.* For optional part, try using the fact that $a_n$ will be bounded and $a_n - L = (\sqrt{a_n} - \sqrt{L})(\sqrt{a_n} + \sqrt{L})$.

10. Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent if and only if both the sub-sequences $\{a_2n\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

**Solution.** The implication " $\implies$ " is obvious. For the converse, suppose both $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ converge to $\ell$. Let $\epsilon > 0$ be given. Choose $n_1,\ n_2 \in \mathbb{N}$ such that $|a_{2n} - \ell| < \epsilon$ for all $n \geq n_1$ and $|a_{2n+1} - \ell| < \epsilon$ for all $n \geq n_2$. Let $n_0 = max\{n_1, n_2\}$. Then $|a_n - \ell| < \epsilon$ for all $n \geq 2n_0 + 1$.

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# Solutions to Tutorial Sheet 2

2. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\alpha} f(x)$ exists for $\alpha \in \mathbb{R}$. Show that

$$\lim_{h\to 0}[f(\alpha+h) - f(\alpha-h)] = 0.$$

Analyse the converse.

**Solution.** Suppose $\lim_{x\to\alpha} f(x) = L$. Then $\lim_{h\to 0} f(\alpha+h) = L$, and since

$$|f(\alpha+h) - f(\alpha-h)| \le |f(\alpha+h) - L| + |f(\alpha-h) - L|$$

it follows that

$$\lim_{h\to 0} |f(\alpha+h) - f(\alpha-h)| = 0.$$

The converse is *false*. For a counter-example, consider $\alpha = 0$ and

$$f(x) = \begin{cases} 1 & x = 0 \\ \frac{1}{|x|} & x \neq 0 \end{cases}$$

3. Discuss the continuity of the following functions:

(i) $f(x) = \sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

**Solution.** Continuous everywhere except at $x = 0$. To see that $f$ is not continuous at $x = 0$, consider the sequences $\{x_n\}_{n\ge 1}$, $\{y_n\}_{n\ge 1}$ where

$$x_n := \frac{1}{n\pi} \text{ and } y_n := \frac{1}{2n\pi + \frac{\pi}{2}}.$$

Note that both $x_n$, $y_n \to 0$, but $f(x_n) \to 0$ and $|f(y_n)| \to 1$.

Since there exists a finite (equal to 1 in absolute value) between two infinitesimally close values of $x$, the function $f$ is discontinuous at $x = 0$. The value that the function converges to should be exactly the same for *any* choice of sequence converging to the point of concern (here $x = 0$).

(ii) $f(x) = x\sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

**Solution.** Continuous everywhere. For proving the continuity of $f$ at $x = 0$, note that $|f(x)| \le |x|$, and $f(0) = 0$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If $f$ is continuous at 0, show that $f$ is continuous at every $c \in \mathbb{R}$.

**Optional.** Show that the function $f$ satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$.

**Solution.** Taking $x = y = 0$, we get $f(0+0) = 2f(0)$ so that $f(0) = 0$. By the assumption of the continuity of $f$ at 0, $\lim_{x\to 0} f(x) = 0$. Thus,

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0}[f(c) + f(h)] = f(c)$$

showing that $f$ is continuous at $x = c$.

*Hint.* For the optional part, first verify the equality for all $k \in \mathbb{Q}$ and then use the continuity of $f$ to establish it for all $k \in \mathbb{R}$.

5. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that $f$ is differentiable on $\mathbb{R}$. Is $f'$ a continuous function?

**Solution.** Clearly, $f$ is differentiable for all $x \neq 0$ and the derivative is

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), x \neq 0.$$

Also,

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = 0.$$

Clearly, $f'$ is continuous at any $x \neq 0$. However $\lim_{x\to 0} f'(x)$ does not exist. Indeed, for any $\delta > 0$, we can choose $n \in \mathbb{N}$ such that $x := 1/n\pi$, $y := 1/(n+1)\pi$ are in $(-\delta, \delta)$, but $|f'(x) - f'(y)| = 2$.

7. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that

$$\lim_{h\to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

exists and equals $f'(c)$. Is the converse true? [Hint: Consider $f(x) = |x|$.]

**Solution.** For the first part, observe that

$$\begin{aligned}\lim_{h\to 0^+} \frac{f(c+h) - f(c-h)}{2h} &= \lim_{h\to 0^+} \frac{1}{2}\left[\frac{f(c+h) - f(c)}{h} + \frac{f(c-h) - f(c)}{-h}\right] \\ &= \frac{1}{2}[f'(c) + f'(c)] = f'(c)\end{aligned}$$

The converse is *false*. Consider, for example, $f(x) = |x|$ and $c = 0$.

9. Using the theorem on derivatives of inverse function, compute the derivative of

(i) $cos^{-1}x$, $-1 < x < 1$

**Solution.** Let $f(x) = cos(x)$. Then $f'(x) = -\sin(x) \neq 0$ for $x \in (0, \pi)$.

Thus $g(y) = f^{-1}(y) = cos^{-1}(y)$, $-1 < y < 1$ is differentiable and

$$g'(y) = \frac{1}{f'(x)} \text{ where } x \text{ is such that } f(x) = y.$$

Therefore,

$$g'(y) = \frac{-1}{\sin(x)} = \frac{-1}{\sqrt{1 - cos^2(x)}} = \frac{-1}{\sqrt{1 - y^2}}.$$

(ii) $cosec^{-1}x$, $|x| > 1$

**Solution.** Note that

$$cosec^{-1}(x) = \sin^{-1}\frac{1}{x} \text{ for } |x| > 1$$

Since

$$\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1 - x^2}} \text{ for } |x| < 1,$$

one has, by the chain rule

$$\frac{d}{dx}cosec^{-1}(x) = \frac{1}{\sqrt{1-\frac{1}{x^2}}}\left(\frac{-1}{x^2}\right), |x| > 1.$$

10. Compute $\frac{dy}{dx}$, given

$$y = f\left(\frac{2x-1}{x+1}\right) \text{ and } f'(x) = \sin(x^2)$$

**Solution.** By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= f'\left(\frac{2x-1}{x+1}\right)\frac{d}{dx}\left(\frac{2x-1}{x+1}\right) \\ &= \sin\left(\frac{2x-1}{x+1}\right)^2\left[\frac{3}{(x+1)^2}\right] \\ &= \frac{3}{(x+1)^2}\sin\left(\frac{2x-1}{x+1}\right)^2\end{aligned}$$

11. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous every where and is differentiable everywhere except at 2 points.

**Solution.** Consider $f(x); = |x| + |1-x|$ for $x \in \mathbb{R}$.

12. Let $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

Show that $f$ is discontinuous at every $c \in \mathbb{R}$.

**Solution.** For $c \in \mathbb{R}$, select a sequence $\{a_n\}_{n\geq 1}$ of rational numbers and a sequence $\{b_n\}_{n\geq 1}$ of irrational numbers, both converging to $c$. Then $\{f(a_n)\}_{n\geq 1}$ converges to 1 while $\{f(b_n)\}_{n\geq 1}$ converges to 0, showing that limit of $f$ at $c$ does not exist.

15. Let $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$. Show that the following are equivalent:

(i) $f$ is differentiable at $c$

(ii) There exists $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h\to 0} \epsilon_1(h) = 0$ and

$$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta, \delta)$$

(iii) There exists $\alpha \in \mathbb{R}$ such that

$$\lim_{h\to 0}\left(\frac{|f(c+h) - f(c) - \alpha h|}{|h|}\right) = 0$$

**Solution.** To prove equivalence, we need to prove (i) $\iff$ (ii) $\iff$ (iii). We can prove it in a cyclic manner, as (i) $\implies$ (ii), (ii) $\implies$ (iii) and (iii) $\implies$ (i).

(i) $\implies$ (ii): Choose $\delta > 0$ such that $(c-\delta, c+\delta) \subset (a,b)$. Take $\alpha = f'(c)$ and

$$\epsilon_1(h) = \begin{cases} \frac{f(c+h)-f(c)-\alpha h}{h} & h \neq 0 \\ 0 & h = 0 \end{cases}$$

(ii) $\implies$ (iii): $\lim\limits_{h\to 0} \frac{|f(c+h)-f(c)-\alpha h|}{|h|} = \lim\limits_{h \to 0} |\epsilon_1(h)| = 0$

(iii) $\implies$ (i): $\lim\limits_{h\to 0} \left| \frac{f(c+h)-f(c)}{h} - \alpha \right| = 0 \implies \lim\limits_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists and is equal to $\alpha$.

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# Solutions to Tutorial Sheet 3

1. Show that the cubic $x^3 - 6x + 3$ has all roots real.

   **Solution.** $f(x) = x^3 - 6x + 3$ has stationary points at $x = \pm\sqrt{2}$. Note that $f(-\sqrt{2}) = 4\sqrt{2} + 3 > 0$ and $f(\sqrt{2}) = 3 - 4\sqrt{2} < 0$. Therefore, $f$ has a root in $(-\sqrt{2}, \sqrt{2})$. Also, $\lim_{x\to-\infty} f(x) \to -\infty$ and so $f$ has a root in $(-\infty, -\sqrt{2})$ and $\lim_{x\to\infty} f(x) \to \infty$ implies $f$ has a root in $(\sqrt{2}, \infty)$. Since $f$ has at host 3 roots, all the roots are real.

4. Consider the cubic $f(x) = x^3 + px + q$, where $p$ and $q$ are real numbers. If $f$ has three distinct real roots, show that $4p^3 + 27q^2 < 0$ by proving the following:

   (i) $p < 0$.

   **Solution.** Since $f$ has 3 distinct roots, say $r_1 < r_2 < r_3$, by Rolle's theorem, $f'(x)$ has at least 2 real roots, say $x_1$ and $x_2$ such that $r_1 < x_1 < r_2$ and $r_2 < x_2 < r_3$. Since $f'(x) = 3x^2 + p$, this implies that $p < 0$, so that 2 real roots exist.

   (ii) $f$ has maximum/minimum at $\pm\sqrt{-p/3}$.

   **Solution.** Solving $f'(x) = 0$, we get $x_1 = -\sqrt{-p/3}$ and $x_2 = \sqrt{-p/3}$. Now $f''(x_1) = 6x_1 < 0$ and so $f$ has a local maxima at $x = x_1$. Similarly, $f$ has a local minima at $x = x_2$.

   (iii) The maximum/minimum values are of opposite signs.

   **Solution.** Since $f'(x)$ is negative between its roots $x_1$ and $x_2$ and $f$ has a root $r_2$ in $(x_1, x_2)$, we must have $f(x_1) > 0$ and $f(x_2) < 0$. Further,

   $$f(x_1) = q + \sqrt{\frac{-4p^3}{27}}, f(x_2) = q - \sqrt{\frac{-4p^3}{27}}$$

   so that

   $$\frac{4p^3 + 27q^2}{27} = f(x_1)f(x_2) < 0.$$

5. Use the MVT to prove $|\sin a - \sin b| \leq |a - b$ for all $a, b \in \mathbb{R}$.

   **Solution.** For some $c$ between $a$ and $b$, one has, by MVT,

   $$\left|\frac{\sin a - \sin b}{a - b}\right| = |\cos c| \leq 1$$

7. Let $a > 0$ and $f$ be continuous on $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.

   **Optional.** Show that under the given conditions, in fact $f(x) = x$ for every $x$.

   **Solution.** By Lagrange's MVT, there exists $c_1 \in (-a, 0)$ and there exists $c_2 \in (0, a)$ such that

   $$f(0) - f(-a) = a \times f'(c_1) \text{ and } f(a) - f(0) = a \times f'(c_2)$$

Using the given conditions, we obtain

$$f(0) + a \leq a \text{ and } a - f(0) \leq a$$

which implies $f(0) = 0$.

*Hint.* For the optional part, consider $g(x) = f(x) - x$, $x \in [-a, a]$.

**Solution.** Since $g'(x) = f'(x) - 1 \leq 0$, $g$ is decreasing over $[-a, a]$. As $g(-a) = g(a) = 0$, we have $g \equiv 0$.

8. In each case, find a function $f$ which satisfies all the given conditions, or else show that no such function exists.

(i) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 1$

**Solution.** No such function exists in view of Rolle's theorem on $[0, 1]$.

(ii) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f'(1) = 2$.

**Solution.** $f(x) = x + \frac{x^2}{2}$

(iii) $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 100$ for all $x > 0$

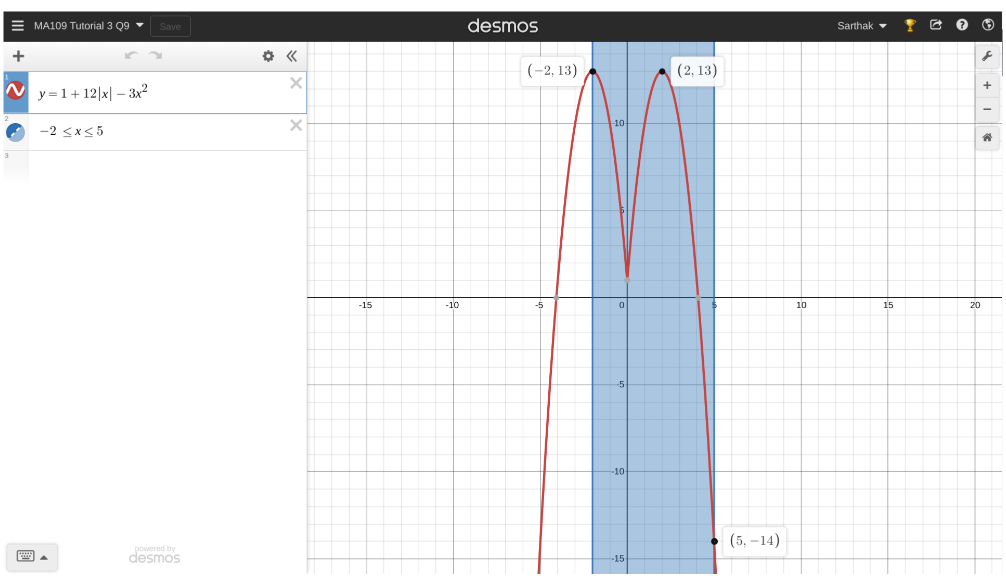
**Solution.** $f''(x) \geq 0 \implies f'$ increasing. As $f'(0) = 1$, by Lagrange's MVT we have $f(x) - f(0) \geq x$ for $x > 0$. Hence $f$ with the required properties cannot exist.

(iv) $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1$, $f(x) \leq 1$ for all $x < 0$

**Solution.**

$$f(x) = \begin{cases} \frac{1}{1-x} & x \leq 0 \\ 1 + x + x^2 & x > 0 \end{cases}$$

9. Let $f(x) = 1 + 12|x| - 3x^2$. Find the absolute maximum and the absolute minimum of $f$ on [-2,5]. Verify it from the sketch of the curve $y = f(x)$ on [-2,5].



**Solution.** The points to check are the end points $x = -2$ and $x = 5$, the point of non-differentiability $x = 0$ and the stationary point $x = 2$. The values of $f$ at these points are given by

$$f(-2) = f(2) = 13, f(0) = 1, f(5) = -14.$$

Thus, global maximum is 13 at $x = \pm 2$ and global minimum is -14 at $x = 5$.

10. A window is to be made in the form of a rectangle surmounted by a semicircular portion with diameter equal to the base of the rectangle. The rectangular portion is to be of clear glass and the semicircular portion is to be of colored glass admitting only half as much light per square foot as the clear glass. If the total perimeter of the window frame is to be $p$ feet, find the dimensions of the window which will admit the maximum light.

**Solution.** Let the dimensions of the rectangular portion be $l$ and $b$. Then the radius of the semicircular portion will be $r = b/2$. Then total perimeter is given by

$$p = 2l + b + \pi r = 2l + \left(1 + \frac{\pi}{2}\right) b.$$

Assume that clear glass allows 1 unit of light per square foot. Then the total light admitted by the window is given as

$$L = l \times b \times 1 + \frac{\pi r^2}{2} \times \frac{1}{2} = lb + \frac{\pi}{16} b^2$$

Using the previous relation, we can eliminate $l$, and get

$$L = \frac{\pi}{16} b^2 + b \left( \frac{p - \left(1 + \frac{\pi}{2}\right) b}{2} \right) = b^2 \left( -\frac{3\pi + 8}{16} \right) + b \left( \frac{p}{2} \right)$$

To maximise $L$ with respect to $b$, we set $\frac{dL}{db}$ to 0. This gives us

$$2b \left( -\frac{3\pi + 8}{16} \right) + \frac{p}{2} = 0$$

Hence the dimensions to maximise admitted light are

$$b = \frac{4p}{3\pi + 8} \text{ and } l = \frac{(\pi + 4)p}{2(3\pi + 8)}$$

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# Solutions to Tutorial Sheet 4

5. Let $f(x) = 1$ if $x \in [0, 1]$ and $f(x) = 2$ if $x \in (1, 2]$. Show from the first principles that $f$ is Riemann integrable on $[0, 2]$ and find $\int_0^2 f(x)dx$.

**Solution.** The given function is integrable as it is monotone. Let $P_n$ be the partition of $[0, 2]$ into $2 \times 2^n$ equal parts. Then $U(P_n, f) = 3$ and

$$L(P_n, f) = 1 + 1 \times \frac{1}{2^n} + 2 \times \frac{2^n - 1}{2^n} \to 3$$

as $n \to \infty$. Thus, $\int_0^2 f(x)dx = 3$.

6. (a) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\int_a^b f(x)dx \geq 0$. Further, if $f$ is continuous and $\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

**Solution.** $f(x) \geq 0 \implies U(P, f) \geq 0$, $L(P, f) \geq 0 \implies \int_a^b f(x)dx \geq 0$. Suppose, moreover, $f$ is continuous and $\int_a^b f(x)dx = 0$. Assume $f(c) > 0$ for some $c \in [a, b]$. Then $f(x) > \frac{f(c)}{2}$ in a $\delta$-nbhd of $c$ for some $\delta > 0$. This implies that

$$U(P, f) > \delta \times \frac{f(c)}{2}$$

for any partition $P$, and hence, $\int_a^b f(x)dx \geq \delta\frac{f(c)}{2} > 0$, a contradiction.

(b) Give an example of a Riemann integrable function on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)dx = 0$, but $f(x) \neq 0$ for some $x \in [a, b]$.

**Solution.** On $[0, 1]$ take

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

7. Evaluate $\lim_{n\to\infty} S_n$ by showing that $S_n$ is an approximate Riemann sum for a suitable function over a suitable interval:

(i) $S_n = \frac{1}{n^{5/2}} \sum_{i=1}^{n} i^{3/2}$

**Solution.** $S_n = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^{3/2} \to \int_0^1 x^{3/2}dx = \frac{2}{5}$

(iii) $S_n = \sum_{i=1}^{n} \frac{1}{\sqrt{in + n^2}}$

**Solution.** $S_n = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{1 + \frac{i}{n}}} \to \int_0^1 \frac{dx}{\sqrt{1+x}} = 2(\sqrt{2} - 1)$

(iv) $S_n = \frac{1}{n} \sum_{i=1}^{n} \cos \frac{i\pi}{n}$

**Solution.** $S_n = \frac{1}{n} \sum_{i=1}^{n} \cos \frac{i\pi}{n} \to \int_0^1 \cos \pi x = 0$

(v) $S_n = \frac{1}{n} \left\{ \sum_{i=1}^{n} \left(\frac{i}{n}\right) + \sum_{i=n+1}^{2n} \left(\frac{i}{n}\right)^{3/2} + \sum_{i=2n+1}^{3n} \left(\frac{i}{n}\right)^2 \right\}$

**Solution.** $S_n \to \int_0^1 x dx + \int_1^2 x^{3/2} dx + \int_2^3 x^2 dx = \frac{1}{2} + \frac{2}{5}(4\sqrt{2} - 1) + \frac{19}{3}$

8. (b) Compute $\frac{dF}{dx}$, if for $x \in \mathbb{R}$:

(i) $F(x) = \int_1^{2x} \cos(t^2) dt$

**Solution.** $F'(x) = \cos((2x)^2) \times 2 = 2\cos(4x^2)$

(ii) $F(x) = \int_0^{x^2} \cos(t) dt$

**Solution.** $F'(x) = \cos(x^2) \times 2x = 2x \cos(x^2)$

9. Let $p$ be a real number and let $f$ be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for all $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number $a$.

(*Hint:* Consider $F(a) = \int_a^{a+p} f(t)dt$, $a \in \mathbb{R}$).

**Solution.** Define $F(x) = \int_x^{x+p} f(t)dt$, $x \in \mathbb{R}$. Then $F'(x) = f(x+p) - f(x) = 0$ for every $x$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $\lambda \in \mathbb{R}$, $\lambda \neq 0$. For $x \in \mathbb{R}$, let

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x - t) dt.$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ for all $x \in \mathbb{R}$ and $g(0) = g'(0) = 0$.

**Solution.** Write $\sin \lambda(x-t)$ as $\sin(\lambda x)\cos(\lambda t) - \cos(\lambda x)\sin(\lambda t)$ in the integrand, take terms in $x$ outside the integral, evaluate $g'(x)$ and $g''(x)$, and simplify to show LHS = RHS. From the expressions for $g(x)$ and $g'(x)$ it should be clear that $g(0) = g'(0) = 0$.

**Alternate.** The problem can also be solved by appealing to the following theorem:

**Theorem A.** Let $h(t, x)$ and $\dfrac{\partial h}{\partial x}(t, x)$ be continuous functions of $t$ and $x$ on the rectangle $[a, b] \times [c, d]$. Let $u(x)$ and $v(x)$ be differentiable functions of $x$ on $[c, d]$ such that, for each $x \in [c, d]$, the points $(u(x), x)$ and $(v(x), x)$ belong to $[a, b] \times [c, d]$. Then

$$\frac{d}{dx} \int_{u(x)}^{v(x)} h(t, x) dt = \int_{u(x)}^{v(x)} \frac{\partial h}{\partial x}(t, x) dt - u'(x) h(u(x), x) + v'(x) h(v(x), x).$$

Consider now

$$g(x) = \frac{1}{\lambda} \int_0^x f(t) \sin \lambda(x - t) dt.$$

Let $h(t, x) = \dfrac{1}{\lambda} f(t) \sin \lambda(x - t)$, $u(x) = 0$ and $v(x) = x$. Then it follows from Theorem A that

$$g'(x) = \int_0^x f(t) \cos \lambda(x - t) dt.$$

Again, applying Theorem A, we have

$$g''(x) = -\lambda \int_0^x f(t) \sin \lambda(x - t) + f(x).$$

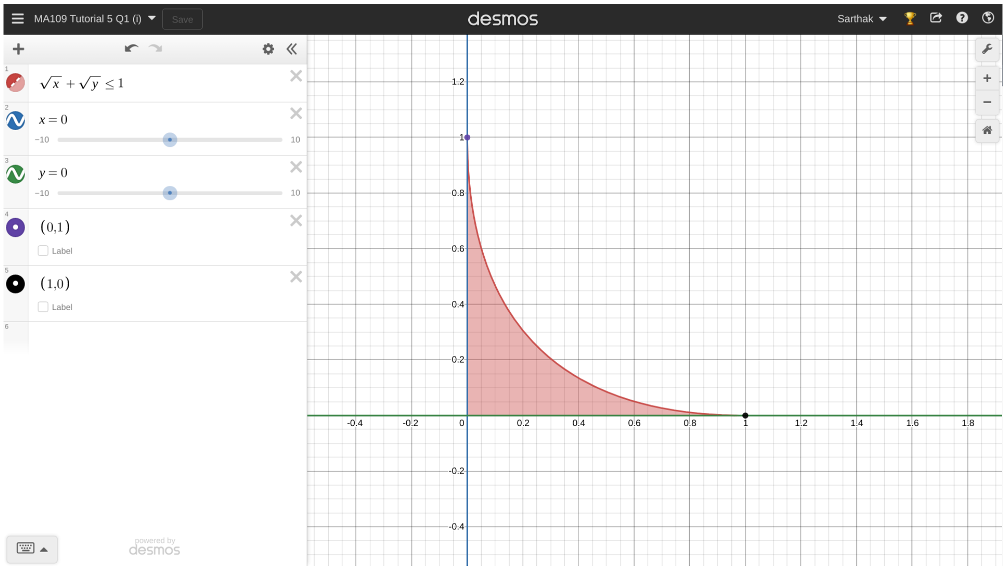
Thus $g''(x) + \lambda^2 g(x) = f(x)$. $g(0) = g'(0) = 0$ is obvious from the expressions for $g(x)$ and $g'(x)$.

Credits: MA105 Tutorial Solutions (2014)

# Solutions to Tutorial Sheet 5

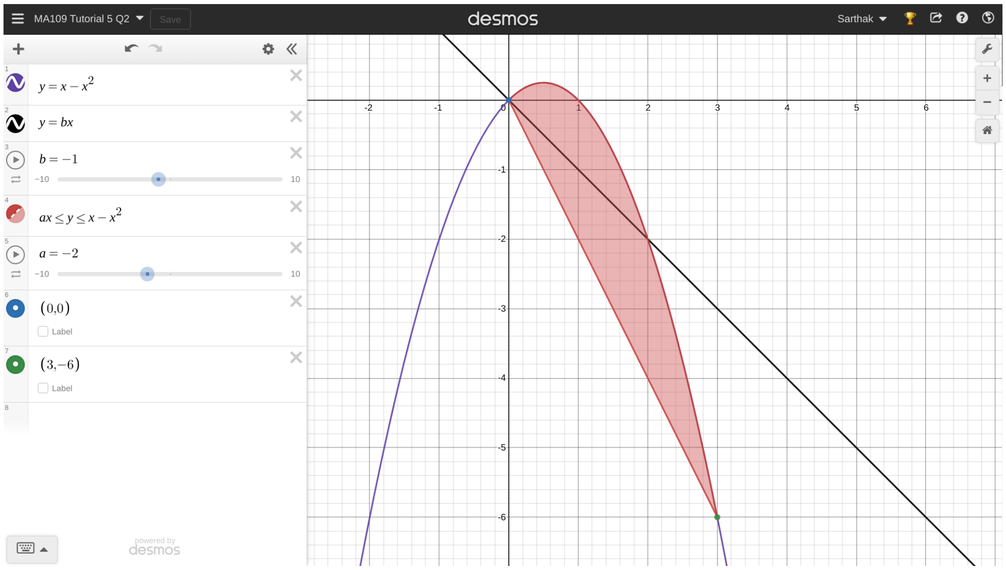
1. Find the area of the region bounded by the given curves in each of the following cases:

(i) $\sqrt{x} + \sqrt{y} = 1$, $x = 0$ and $y = 0$.




**Solution.** $\displaystyle\int_0^1 y dx = \int_0^1 \left(1 + x - 2\sqrt{x}\right) dx = 1 + \frac{1}{2} - 2 \times \frac{2}{3} = \frac{1}{6}.$

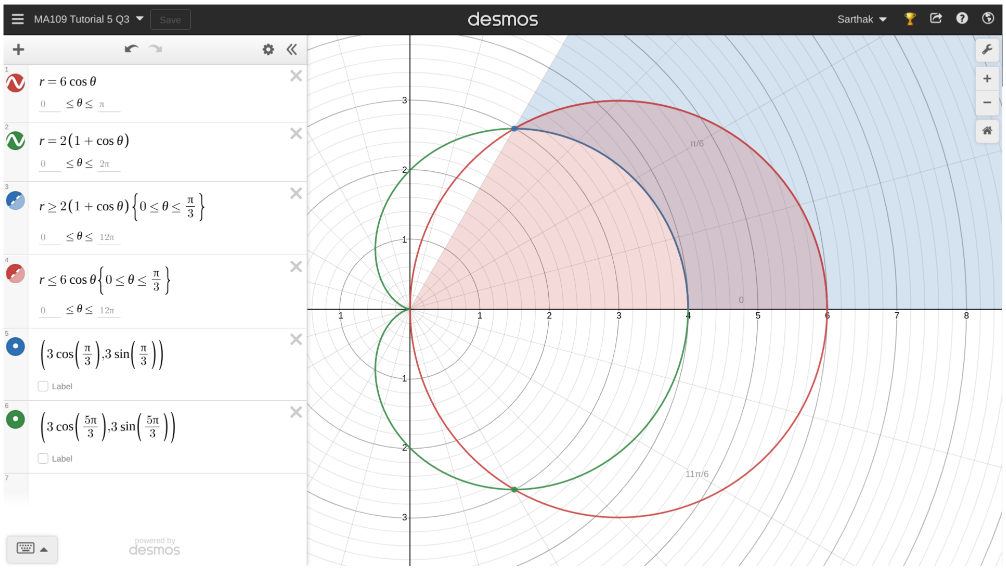
2. Let $f(x) = x - x^2$ and $g(x) = ax$. Determine $a$ so that the region above the graph of $g$ and below the graph of $f$ has area $\dfrac{9}{2}$.



**Solution.** $\left|\int_0^{1-a} (x - x^2 - ax)\, dx\right| = \left|\int_0^{1-a} ((1-a)x - x^2)\, dx\right| = \frac{9}{2} \implies \left|\frac{(1-a)^3}{6}\right| = \frac{9}{2}$

$$a = -2, 4.$$

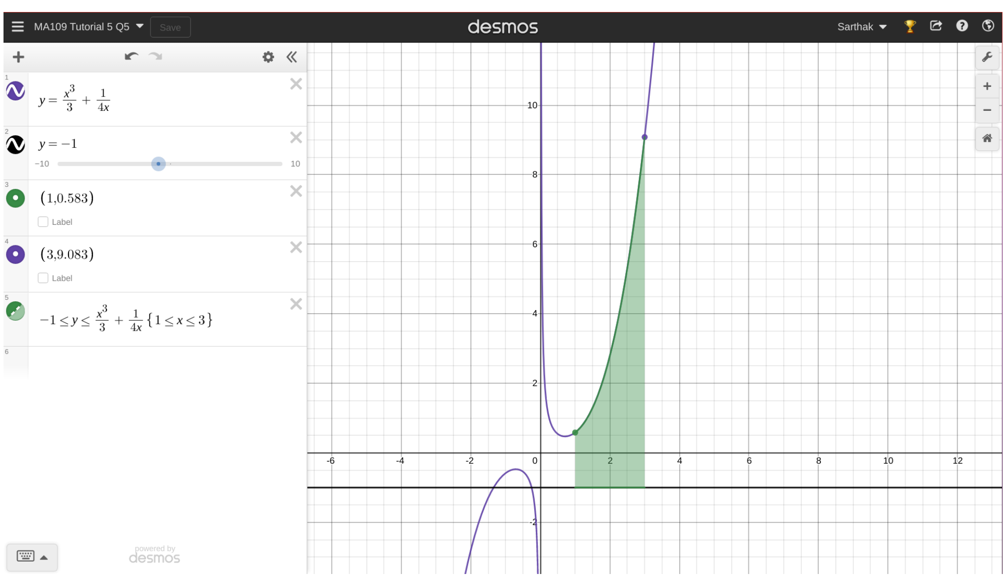
3. Find the area of the region inside the circle $r = 6a\cos\theta$ and outside the cardioid $r = 2a(1+\cos\theta)$.




**Solution.** Required area $= 2 \times \int_0^{\pi/3} \frac{1}{2}(r_2^2 - r_1^2)d\theta = 4a^2 \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)d\theta = 4\pi a^2.$

5. For the following curve, find the arc length as well as the area of the surface generated by revolving it about the line $y = -1$:

$$y = \frac{x^3}{3} + \frac{1}{4x},\ 1 \le x \le 3$$



**Solution.** $\dfrac{dy}{dx} = x^2 + \left(-\dfrac{1}{4x^2}\right) \implies \dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} = \sqrt{1 + x^4 + \dfrac{1}{16x^4} - \dfrac{1}{2}} = x^2 + \dfrac{1}{4x^2}.$

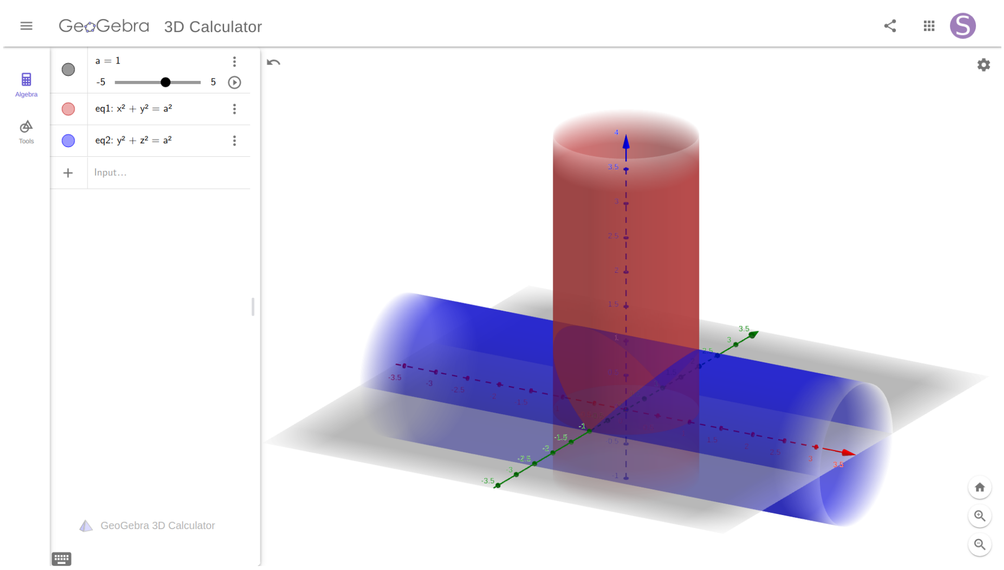
Therefore, the arc length is given by,

$$\int_1^3 \left(x^2 + \frac{1}{4x^2}\right) dx = \left[\frac{x^3}{3} - \frac{1}{4x}\right]_1^3 = \frac{53}{6}.$$

The surface area is,

$$S = \int_1^3 2\pi(y+1)\frac{ds}{dx}dx = \int_1^3 2\pi\left(\frac{x^3}{3} + \frac{1}{4x} + 1\right)\left(x^2 + \frac{1}{4x^2}\right)dx$$
$$= 2\pi\left[\frac{x^6}{18} + \frac{x^3}{3} + \frac{x^2}{6} - \frac{1}{32x^2} - \frac{1}{4x}\right]_1^3 = \frac{1823}{18}\pi$$

7. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $y^2 + z^2 = a^2$.




**Solution.** In the first octant, the sections perpendicular to the $y$-axis are squares with

$$0 \le x \le \sqrt{a^2 - y^2},\ 0 \le z \le \sqrt{a^2 - y^2},\ 0 \le y \le a.$$

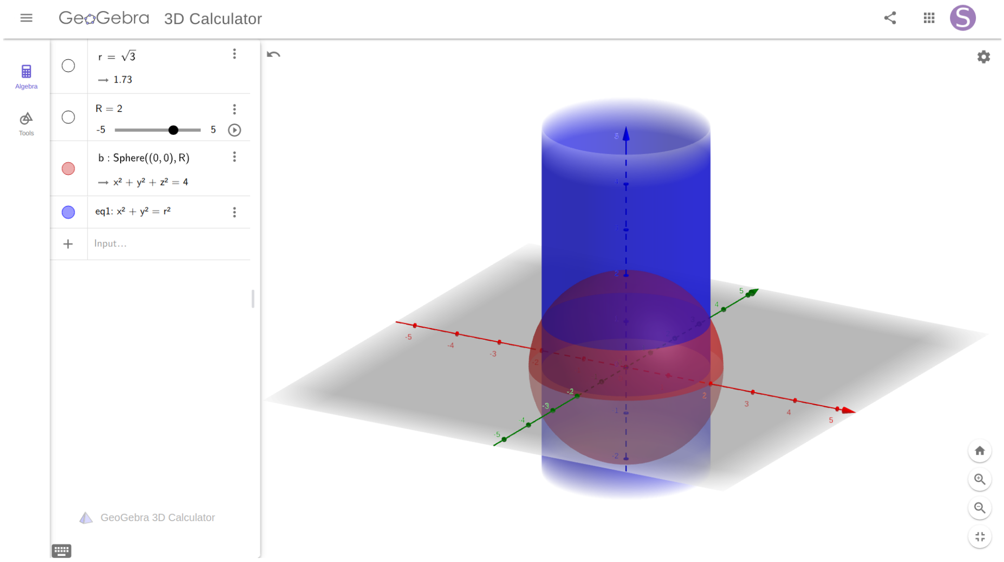
Since the squares have sides of length $\sqrt{a^2 - y^2}$, the area of the cross-section at $y$ is $A(y) = 4(a^2 - y^2)$. Thus the required volume is

$$\int_{-a}^{a} A(y)dy = 8\int_0^a (a^2 - y^2)dy = \frac{16a^3}{3}.$$

8. A fixed line $L$ in 3-space and a square of side $r$ in a plane perpendicular to $L$ are given. One vertex of the square is on $L$. As this vertex moves a distance $h$ along $L$, the square turns through a full revolution with $L$ as the axis. Find the volume of the solid generated by this motion.

**Solution.** Let the line be along $z$-axis, $0 \le z \le h$. For any fixed $z$, the section is a square of area $r^2$. Hence the required volume is $\int_0^h r^2 dz = r^2 h$.

10. A round hole of radius $r = \sqrt{3}$ cm is bored through the center of a solid ball of radius $R = 2$ cm. Find the volume cut out.




**Solution.** Required volume = Volume of the sphere − Volume generated by revolving the shaded region around the $y$-axis.

**Washer Method:** Integrating $x$ as a function of $y$ (using horizontal solid circular washers)

$$\frac{32}{3}\pi - \left[\int_{-1}^{1} \pi x^2 dy - 2 \times \pi \left(\sqrt{3}\right)^2\right] = \frac{32}{3}\pi - 2\pi\left[\int_0^1 (4 - y^2)dy - 3\right] = \frac{32}{3}\pi - 2\pi\left[\frac{11}{3} - 3\right] = \frac{28}{3}\pi.$$

**Shell Method:** Integrating $y$ as a function of $x$ (using vertical hollow cylindrical shells)

$$\frac{32}{3}\pi - \int_{\sqrt{3}}^{2} 2\pi x \times 2y dx = \frac{32}{3}\pi - 4\pi \int_{\sqrt{3}}^{2} x\sqrt{4 - x^2} dx = \frac{32}{3}\pi - 4\pi\frac{1}{3} = \frac{28}{3}\pi.$$

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# Solutions to Tutorial Sheet 6

2. Describe the level curves and the contour lines for the following functions corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$:

(i) $f(x, y) = x - y$

**Solution.** A level curve corresponding to any of the given values of $c$ is the straight line $x - y = c$ in the $xy$-plane. A contour line corresponding to any of the given values of $c$ is the same line shifted to the plane $z = c$ in $\mathbb{R}^3$.

(ii) $f(x, y) = x^2 + y^2$

**Solution.** Level curves do not exist for $c = -3, -2, -1$. The level curve corresponding to $c = 0$ is the point $(0, 0)$. The level curves corresponding to $c = 1, 2, 3, 4$ are concentric circles centered at the origin in the $xy$-plane. Contour lines corresponding to $c = 1, 2, 3, 4$ are the cross-sections in $\mathbb{R}^3$ of the paraboloid $z = x^2 + y^2$ by the plane $z = c$, i.e., circles in the plane $z = c$ centered at $(0, 0, c)$.

(iii) $f(x, y) = xy$

**Solution.** For $c = -3, -2, -1$, level curves are rectangular hyperbolas $xy = c$ in the $xy$-plane with branches in the second and fourth quadrant. For $c = 1, 2, 3, 4$, level curves are rectangular hyperbolas $xy = c$ in the $xy$-plane with branches in the first and third quadrant. For $c = 0$, the corresponding level curve (resp. the contour line) is the union of the $x$-axis and the $y$-axis in the $xy$-plane (resp. in the $xyz$-space). A contour line corresponding to a non-zero $c$ is the cross-section of the hyperboloid $z = xy$ by the plane $z = c$, i.e., a rectangular hyperbola in the plane $z = c$.

3. Using definition, examine the following functions for continuity at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero:

(i) $\dfrac{x^3 y}{x^6 + y^2}$

**Solution.** Discontinuous at $(0, 0)$ (check $\lim_{(x,y)\to(0,0)} f(x, y)$ using $y = mx^3$).

(ii) $xy\dfrac{x^2 - y^2}{x^2 + y^2}$

**Solution.** Continuous at $(0, 0)$ (since $\left| xy\dfrac{x^2 - y^2}{x^2 + y^2} \right| \leq |xy|$).

(iii) $||x| - |y|| - |x| - |y|$

**Solution.** Continuous at $(0, 0)$ (since $|f(x, y)| \leq 2(|x| + |y|) \leq 4\sqrt{x^2 + y^2}$).

6. Examine the following functions for the existence of partial derivatives at $(0, 0)$. The expressions below give the value at $(x, y) \neq (0, 0)$. At $(0, 0)$, the value should be taken as zero:

(i) $xy\dfrac{x^2 - y^2}{x^2 + y^2}$

**Solution.** $f_x(x,y) = y\left(1 - \frac{2y^4}{(x^2+y^2)^2}\right)$ and $f_y(x,y) = x\left(\frac{2x^4}{(x^2+y^2)^2} - 1\right)$, hence $|f_x(x,y)| \le |y|$ and $|f_y(x,y)| \le |x|$ (since $0 \le \frac{y^4}{(x^2+y^2)^2} \le 1$ and $0 \le \frac{x^4}{(x^2+y^2)^2} \le 1$). So $f_x(0,0) = 0 = f_y(0,0)$.

(ii) $\frac{\sin^2(x+y)}{|x|+|y|}$

**Solution.** $f_x(0,0) = \lim_{h\to 0} \frac{\sin^2(h)/|h|}{h} = \lim_{h\to 0} \frac{\sin^2(h)}{h|h|}$ does not exist (because LHL $\neq$ RHL).

Similarly, $f_y(0,0)$ does not exist.

7. Let $f(0,0) = 0$ and

$$f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2} \text{ for } (x,y) \neq (0,0).$$

Show that $f$ is continuous at $(0,0)$, and the partial derivatives of $f$ exist but are not bounded in any disc (howsoever small) around $(0,0)$.

**Solution.** $|f(x,y)| \le x^2+y^2 \implies f$ is continuous at $(0,0)$. Now,

$$f_x = 2x\left(\sin\left(\frac{1}{x^2+y^2}\right) - \frac{1}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)\right)$$

It is easily checked that $f_x(0,0) = f_y(0,0) = 0$. The function $2x\sin\left(\frac{1}{x^2+y^2}\right)$ is bounded in any disc centered at $(0,0)$, while $\frac{2x}{x^2+y^2}\cos\left(\frac{1}{x^2+y^2}\right)$ is unbounded in any such disc (consider for example $(x,y) = \left(\frac{1}{\sqrt{n\pi}}, 0\right)$ for $n$ a large positive integer). Thus $f_x(0,0)$ is unbounded in any disc around $(0,0)$.

8. Let $f(0,0) = 0$ and

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y) & x \neq 0, y \neq 0 \\ x\sin(1/x) & x \neq 0, y = 0 \\ y\sin(1/y) & y \neq 0, x = 0 \end{cases}$$

Show that none of the partial derivatives of $f$ exist at $(0,0)$ although $f$ is continuous at $(0,0)$.

**Solution.** $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \sin\frac{1}{h}$ does not exist. Similarly $f_y(0,0)$ does not exist. Clearly, $f$ is continuous at $(0,0)$.

9. Examine the following functions for the existence of directional derivatives and differentiability at $(0,0)$. The expressions below give the value at $(x,y) \neq (0,0)$. At $(0,0)$, the value should be taken as zero:

(i) $xy\frac{x^2-y^2}{x^2+y^2}$

**Solution.** Let $\vec{v} = (a,b)$ be any unit vector in $\mathbb{R}^2$. We have

$$(D_{\vec{v}}f)(0,0) = \lim_{h\to 0} \frac{f(h\vec{v})}{h} = \lim_{h\to 0} \frac{f(ha,hb)}{h} = \lim_{h\to 0} \frac{h^2ab\left(\frac{a^2-b^2}{a^2+b^2}\right)}{h} = 0.$$

Therefore $(D_{\vec{v}}f)(0,0)$ exists and equals 0 for every unit vector $\vec{v}$ in $\mathbb{R}^2$. For considering differentiability, note that $f_x(0,0) = (D_{\hat{i}}f)(0,0) = 0 = f_y(0,0) = (D_{\hat{j}}f)(0,0)$. We have then

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk(h^2-k^2)|}{(h^2+k^2)^{3/2}} = 0,$$

since

$$0 \leq \frac{|hk(h^2-k^2)}{(h^2+k^2)^{3/2}} \leq \frac{|hk|}{\sqrt{h^2+k^2}} \leq \sqrt{h^2+k^2}.$$

Thus $f$ is differentiable at $(0,0)$.

(ii) $\dfrac{x^3}{x^2+y^2}$

**Solution.** Note that for any unit vector $\vec{v} = (a,b)$ in $\mathbb{R}^2$, we have

$$(D_{\vec{v}}f)(0,0) = \lim_{h\to 0} \frac{h^3a^3}{h(h^2(a^2+b^2))} = \lim_{h\to 0} \frac{a^3}{(a^2+b^2)} = \frac{a^3}{(a^2+b^2)}.$$

To consider differentiability, note that $f_x(0,0) = 1$, $f_y(0,0) = 0$ and

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - h\times 1 - k\times 0|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|h^3/(h^2+k^2) - h|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk^2|}{(h^2+k^2)^{3/2}}$$

does not exist (consider $k = mh$, and then put $m = 10^{-5}$). Hence $f$ is not differentiable at $(0,0)$.

(iii) $(x^2+y^2)\sin\dfrac{1}{x^2+y^2}$

**Solution.** For any unit vector $\vec{v} \in \mathbb{R}^2$, one has,

$$(D_{\vec{v}}f)(0,0) = \lim_{h\to 0} \frac{h^2(a^2+b^2)\sin\left[\frac{1}{h^2(a^2+b^2)}\right]}{h} = 0.$$

Also,

$$\lim_{(h,k)\to(0,0)} \frac{\left|(h^2+k^2)\sin\left[\frac{1}{h^2+k^2}\right]\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right) = 0.$$

Therefore $f$ is differentiable at $(0,0)$.

10. Let $f(x,y) = 0$ if $y = 0$ and

$$f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2} \text{ if } y \neq 0.$$

Show that $f$ is continuous at $(0,0)$, $(D_{\vec{v}}f)(0,0)$ exists for every vector $\vec{v}$, yet $f$ is not differentiable at $(0,0)$.

**Solution.** $f(0,0) = 0$, $|f(x,y)| \leq \sqrt{x^2+y^2} \implies f$ is continuous at $(0,0)$. Let $\vec{v}$ be a unit vector in $\mathbb{R}^2$. For $\vec{v} = (a,b)$, with $b \neq 0$, one has,

$$(D_{\vec{v}}f)(0,0) = \lim_{h\to 0} \frac{1}{h}\frac{hb}{|hb|}\sqrt{h^2a^2+h^2b^2} = \frac{(\sqrt{a^2+b^2})b}{|b|}.$$

If $\vec{v} = (a, 0)$, then $(D_{\vec{v}} f)(0,0) = 0$. Hence $(D_{\vec{v}} f)(0,0)$ exists for every unit vector $\vec{v} \in \mathbb{R}^2$. Further,

$$f_x(0,0) = 0,\ f_y(0,0) = 1,$$

and

$$\lim_{(h,k)\to(0,0)} \frac{|f(h,k) - 0 - h \times 0 - k \times 1|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{\left|\frac{k}{|k|}\sqrt{h^2+k^2} - k\right|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \left|\frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}}\right|$$

does not exist (consider $h = mk$ and then put $m = 10^{-5}$) so that $f$ is not differentiable at $(0,0)$.

Credits: MA105 Tutorial Solutions (2014)

# Solutions to Tutorial Sheet 7

1. Let $F(x,y,z) = x^2 + 2xy - y^2 + z^2$. Find the gradient of $F$ at $(1,-1,3)$ and the equations of the tangent plane and the normal line to the surface $F(x,y,z) = 7$ at $(1,-1,3)$.

   **Solution.** $(\nabla F)(1,-1,3) = \left(\frac{\partial F}{\partial x}(1,-1,3), \frac{\partial F}{\partial y}(1,-1,3), \frac{\partial F}{\partial z}(1,-1,3)\right) = 4\mathbf{j} + 6\mathbf{k}$.

   The tangent plane to the surface $F(x,y,z) = 7$ at the point $(1,-1,3)$ is given by,

   $$0 \times (x-1) + 4 \times (y+1) + 6 \times (z-3) = 0 \implies 2y + 3z = 7.$$

   The normal line to the surface $F(x,y,z) = 7$ at the point $(1,-1,3)$ is given by $x = 1, 3y - 2z + 9 = 0$.

2. Find $D_{\vec{u}}F(2,2,1)$, where $F(x,y,z) = 3x - 5y + 2z$, and $\vec{u}$ is the unit vector in the direction of the outward normal to the sphere $x^2 + y^2 + z^2 = 9$ at $(2,2,1)$.

   **Solution.** $\vec{u} = \frac{(2,2,1)}{\sqrt{2^2+2^2+1^2}} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = \frac{2}{3}(\mathbf{i}+\mathbf{j})\frac{1}{3}\mathbf{k}$ and $(\nabla F)(2,2,1) = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$. Therefore, $D_{\vec{u}}F(2,2,1) = (\nabla F)(2,2,1) \cdot \vec{u} = \frac{6}{3} - \frac{10}{3} + \frac{2}{3} = -\frac{2}{3}$.

3. Given $\sin(x+y) + \sin(y+z) = 1$, find $\frac{\partial^2 z}{\partial x \partial y}$, provided $\cos(y+z) \neq 0$.

   **Solution.** Given that $\sin(x+y) + \sin(y+z) = 1$ (with $\cos(y+z) \neq 0$).

   You may assume that $z$ is a sufficiently smooth function of $x$ and $y$.

   Differentiating w.r.t. $x$ while keeping $y$ fixed, we get,

   $$\cos(x+y) + \cos(y+z)\frac{\partial z}{\partial x} = 0. \tag{*}$$

   Similarly, differentiating w.r.t. $y$ while keeping $x$ fixed, we get,

   $$\cos(x+y) + \cos(y+z)\left(1 + \frac{\partial z}{\partial y}\right) = 0. \tag{**}$$

   Differentiating $(*)$ w.r.t $y$ we have,

   $$-\sin(x+y) - \sin(y+z)\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} + \cos(y+z)\frac{\partial^2 z}{\partial x \partial y} = 0.$$

   Thus, using $(*)$ and $(**)$, we have,

   $$\begin{aligned}
   \frac{\partial^2 z}{\partial x \partial y} &= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}\right] \\
   &= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\right] \\
   &= \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\left(\frac{\cos^2(x+y)}{\cos^2(y+z)}\right)
   \end{aligned}$$

4. If $f(0,0) = 0$ and

$$f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2} \text{ for } (x,y) \neq (0,0),$$

show that both $f_{xy}$ and $f_{yx}$ exist at $(0,0)$, but they are not equal. Are $f_{xy}$ and $f_{yx}$ continuous at $(0,0)$?

**Solution.** We have,

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k},$$

where (noting that $k \neq 0$),

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = -k \text{ and } f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0.$$

Therefore,

$$f_{xy}(0,0) = \lim_{k\to 0} \frac{-k - 0}{k} = -1 \text{ ; similarly } f_{yx}(0,0) = 1.$$

Thus,

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

By directly computing $f_{xy}$, $f_{yx}$ for $(x,y) \neq (0,0)$, one observes that these are not continuous at $(0,0)$.

5. Show that the following functions have local minima at the indicated points:

(i) $f(x,y) = x^4 + y^4 + 4x - 32y - 7$, $(x_0, y_0) = (-1, 2)$

**Solution.** $f_x(-1,2) = 0 = f_y(-1,2)$, $H_f(-1,2) = \begin{bmatrix} 12 & 0 \\ 0 & 48 \end{bmatrix}$

$D(-1,2) = 12 \times 48 - 0^2 > 0$, $f_{xx}(-1,2) = 12 > 0 \implies (-1,2)$ is a point of local minimum of $f$.

(ii) $f(x,y) = x^3 + 3x^2 - 2xy + 5y^2 - 4y^3$, $(x_0, y_0) = (0,0)$

**Solution.** $f_x(0,0) = 0 = f_y(0,0)$, $H_f(0,0) = \begin{bmatrix} 6 & -2 \\ -2 & 10 \end{bmatrix}$

$D(0,0) = 6 \times 10 - (-2)^2 > 0$, $f_{xx}(0,0) = 6 > 0 \implies (0,0)$ is a point of local minimum of $f$.

6. Analyze the following functions for local maxima, local minima and saddle points:

(i) $f(x,y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$

**Solution.** $f_x = e^{-(x^2+y^2)/2}(2x - x^3 + xy^2)$, $f_y = e^{-(x^2+y^2)/2}(-2y + y^3 - x^2 y)$.

Critical points are $(0,0)$, $(\pm\sqrt{2}, 0)$, $(0, \pm\sqrt{2})$.

$H_f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} \implies (0,0)$ is a saddle point of $f$.

$H_f(\pm\sqrt{2}, 0) = \begin{bmatrix} -4/e & 0 \\ 0 & -4/e \end{bmatrix} \implies (\pm\sqrt{2}, 0)$ are points of local maximum of $f$.

$H_f(0, \pm\sqrt{2}) = \begin{bmatrix} 4/e & 0 \\ 0 & 4/e \end{bmatrix} \implies (0, \pm\sqrt{2})$ are points of local minimum of $f$.

(ii) $f(x,y) = x^3 - 3xy^2$

**Solution.** $f_x = 3x^2 - 3y^2$ and $f_y = -6xy$ imply that $(0,0)$ is the only critical point of $f$. Now,

$$H_f(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus the standard derivative test fails. However, $f(\pm\epsilon, 0) = \pm\epsilon^3$ for any $\epsilon$ so that $(0,0)$ is a saddle point of $f$.

7. Find the absolute maximum and the absolute minimum of

$$f(x,y) = (x^2 - 4x)\cos y \text{ for } 1 \le x \le 3, -\pi/4 \le y \le \pi/4.$$

**Solution.** From $f(x,y) = (x^2 - 4x)\cos y$ $(1 \le x \le 3, -\pi/4 \le y \le \pi/4)$, we have,

$$f_x = (2x-4)\cos y \text{ and } f_y = -(x^2-4x)\sin y.$$

Thus the only critical pointof $f$ is $P = (2,0)$. Note that $f(P) = -4$. Next, $g_\pm(x) \equiv f(x, \pm\frac{\pi}{4}) = \frac{(x^2-4x)}{\sqrt{2}}$ for $1 \le x \le 3$ has $x = 2$ as the only critical point so that we consider $P_\pm = (2, \pm\frac{\pi}{4})$. Note that $f(P_\pm) = \frac{-4}{\sqrt{2}}$. We also need to check $g_\pm(1) = f(1, \pm\frac{\pi}{4})$ $(\equiv f(Q_\pm))$ and $g_\pm(3) = f(3, \pm\frac{\pi}{4})$ $(\equiv f(S_\pm))$. Note that $f(Q_\pm) = \frac{-3}{\sqrt{2}}$ and $f(S_\pm) = -\frac{-3}{\sqrt{2}}$.

Next, consider $h(y) \equiv f(1,y) = -3\cos y$ for $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$. The only critical point of $h$ is $y = 0$. Note that $h(0) = f(1,0)$ $(\equiv f(M)) = -3$. $(h(\pm\frac{\pi}{4})$ is just $f(Q_\pm))$.

Finally, consider $k(y) = f(3,y) = -3\cos y$ for $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$. The only critical point of $k$ is $y = 0$. Note that $k(0) = f(3,0)$ $(\equiv f(T)) = -3$. $(k(\pm\frac{\pi}{4})$ is just $f(S_\pm))$.

Summarizing, we have the following table:

| Points | $P_+$ | $P_-$ | $Q_+$ | $Q_-$ | $S_+$ | $S_-$ | $T$ | $P$ | $M$ |
|---|---|---|---|---|---|---|---|---|---|
| Values | $-\frac{4}{\sqrt{2}}$ | $-\frac{4}{\sqrt{2}}$ | $-\frac{3}{\sqrt{2}}$ | $-\frac{3}{\sqrt{2}}$ | $-\frac{3}{\sqrt{2}}$ | $-\frac{3}{\sqrt{2}}$ | -3 | -4 | -3 |

By inspection one finds that $f_{min} = -4$ attained at $P = (2,0)$ and $f_{max} = \frac{-3}{\sqrt{2}}$ at $Q_\pm = (1, \pm\frac{\pi}{4})$ and at $S_\pm = (3, \pm\frac{\pi}{4})$.

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