MA106 Linear Algebra 2022

Tutorial 6 D1-T4

Kartik Gokhale

Department of Computer Science & Engineering
IIT Bombay

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Problem 1

A matrix is said to be nilpotent if $A^k = 0$ for some $k \in \mathcal{N}$.

- Show that is A is nilpotent, I-A is invertible
- What can you say about the eigenvalues of A? What is the characteristic equation of a nilpotent matrix?
- Given matrices are all nilpotent. Find geometric multiplicities of eigenvalues in each case

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Product of commuting nilpotent matrices is nilpotent. Also show that the result fails if they do not commute.

Problem 1(i)

We need to find a clever guess for the inverse. Consider

$$B = \left(I + A + A^2 \cdots A^{k-1}\right)$$

It can be seen that $(I - A)B = I$ and thus, $A - I$ is invertible.

Problem 1(ii)

$$Ax = \lambda x \implies A^k x = 0 = \lambda^k x \implies \lambda = 0$$

All eigenvalues are 0

Problem 1(iii)

The second matrix is not nilpotent, the other 2 have geometric multiplicities 1 and 2 respectively.

Problem 1(iv)

Consider

$$A^m = 0, B^n = 0, AB = BA$$

WLOG, $n < m$,
Thus

$$(AB)^n = 0$$

and thus, nilpotent
Evident counter example when $n > m$ and matrices do not commute

Problem 2

A matrix is said to be idempotent or a projection if $P^2 = P$

- Show that if P has this property so does I-P
- What can you say about the eigenvalues of P?
- If P is invertible then P=I
- Suppose P is non invertible and $v_1, v_2..v_k$ is a bsis for null space of P. Complete it to a basis $v_1, v_2..v_k, v_{k+1}, ...v_n$. Prove or disprove $Pv_k, Pv_{k+1}, ...Pv_n$ are linearly independent. Can you deduce from this that P is diagonalisable

Problem 2(i)

$$(I - P)^2 = I - 2P + P^2 = I - 2P + P = I - P$$

Problem 2(ii)

$$Px = \lambda x$$

$$P^2 x = Px\lambda x$$

$$\lambda = \lambda^2 \implies \lambda = 0, 1$$

Problem 2(iii)

$$P^{-1}P = I$$
$$P^2 = P$$
$$P^{-1}P^2 = I$$
$$P^{-1}P \times P = I$$
$$P = I$$

Problem 2(iii)

$$P^{-1}P = I$$

$$P^2 = P$$

$$P^{-1}P^2 = I$$

$$P^{-1}P \times P = I$$

$$P = I$$

Problem 2(iv)

Suppose

$$c_{k+1}Pv_{k+!}...c_nPv_n = 0$$

$$P(c_{k+1}v_{k+!}...c_nv_n) = 0$$

So $c_{k+1}v_{k+!}...c_nv_n$ lies in null space of P

$$c_{k+1}v_{k+!}...c_nv_n = a_1v_1 + ..a_nv_n$$

But basis vectors are independent!
Contradiction

Problem 3

Consider the matrix of relection and that of projection, as discussed in Tutorial 0

$$H = I - 2nn^T, H_0 = I - nn^T$$

Find eigenvalues and eigenvectors of H and H_0. Are they diagonalizable? Give reasons. Is H_0 idempotent? Try this out in 2 different ways.

Problem 3

For H:
We obtain eigenvalue -1 with eigenvector n and eigenvalue $+1$ with eigenspace as the vector subspace perpendicular to n. This makes H diagonalisable.
For H_0
We obtain eigenvalue 0 for eigenvector n and eigenvalue $+1$ with eigenspace as the vector subspace perpendicular to n. This makes H diagonalisable. Both are diagonalisable(geometric multiplicity equals algebraic for each eigenvalue) and H_0 is idempotent as $H_0^2 = H_0$

Problem 4

Are the following matrices similar? Why?

$$\begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^3 & 10^{10} & 1 \end{bmatrix}$$

Problem 4

Same eigenvalues and both are diagonalisable.
Diagonalising both gives $diag(2, 1, 3)$ and $diag(3, 2, 1)$ respectively
Thus,

$$\exists T \ s.t. \ T^{-1}AT = B$$

Problem 4

Now suppose its given that determinant of A is one.
We know that the product of all eigenvalues is nothing but the determinant of the matrix. Since our characteristic polynomial is a three dgree one with real coefficients we get two possible cases
Case 1 : All roots real
Suppose none of the roots are 1, Then the product would be $(-1)^3 = (-1) \neq det(A) = 1$ Contradiction
Case 2 : One real root and two complex conjugate roots
Let the real root be α and complex roots be $\beta, \bar{\beta}$, we have

$$\alpha\beta\bar{\beta} = 1$$

$$\implies \alpha = 1$$

Hence Proved

Problem 5

What can you say about the eigenvalues of a skey symmetric matrix? Is a skew-symmetric matrix diagonalisable?

Problem 5

$$MX = \lambda x$$
$$\bar{M} = M$$
$$M\bar{x} = \bar{\lambda}\bar{x}$$

Taking transpose and post multiplying by x

$$(M\bar{x})^T x = (\bar{\lambda}\bar{x})^T x$$

Simplifying the above

$$\bar{\lambda} = -\lambda$$

Problem 5

Diagonalisability depends upon the diagonalisability of the corresponding Hermitian matrix

Let $f : R^3 \mapsto R^3$ be a function such that $f(0) = 0$ and $||f(x) - f(y)|| = ||x - y||$. Is it true that $f(x) = Ax$ for some 3×3 matrix A? What kind of matrix is A?

Problem 6

Utilising

$$||Ax|| = x^T A^T AX$$

We can conclude that

$$A^T A = I$$

Thank you!