MA106 Linear Algebra 2022

Tutorial 5 D1-T4

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Problem 1

Identify the quadratic form and in at least one instance find the directions of the principal axes

- $2xy + 2yz + 2zx = 1$
- $x^2 - 2y^2 + 4z^2 + 6yz = 1$
- $-x^2 - y^2 + 2z^2 + 8xy_4xz + 4yz = 1$

Problem 1(i)

We have the quadratic form as

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We choose A to be symmetric

Problem 1(i)

Thus we have

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We get eigenvalues as -1,-1,2
Thus this is a hyperboloid of 2 sheets

Problem 1(i)

For $\lambda = -1$ eigenvectors are

$$\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

For $\lambda = 2$ eigenvectors are

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix},$$

To get the principal axes all is left to make this an orthogonal set, for this just apply Gram Schmidt procedure on the eigenvectors of $\lambda = -1$ (note different eigenvalues will already have orthogonal eigenvectors)

Problem 1(ii)

Thus we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 3 & 4 \end{bmatrix}$$

We get eigenvalues as $1, 1 \pm 3\sqrt{2}$
Thus this is a hyperboloid of 1 sheets

Problem 1(iii)

Thus we have

$$A = \begin{bmatrix} -1 & 4 & -2 \\ 4 & -1 & 2 \\ -2 & 2 & 2 \end{bmatrix}$$

We get eienvalues as -6,3,3
Thus this is a hyperboloid of 1 sheets

Problem 2

Compute integral

$$\iiint exp(-(2x^2 + 5y^2 + 2z^2 - 4xy - 2xz + 4yz))\, dx\, dy\, dz$$

Problem 2

Thus we have

$$A = \begin{bmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

We get eiegenvalues as 1,1,7
We can thus write A as

$$A = ODO^T$$

where D=diag(1,1,7) and O is orthogonal

Problem 2

In effect we perform the transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow O^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The Jacobian for this is nothing but O^T which has a determinant of 1 or -1 which gives us $|det(J)| = 1$

Problem 2

Hence we are left with

$$\iiint exp(-(X^2 + Y^2 + 7Z^2))\, dX\, dY\, dZ$$

which can easily be computed

Problem 3

Show that $ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz$ factorizes into a product of linear factors (possibly with complex coefficients) iff

$$\left| \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \right| = 0$$

Problem 3

We can write A as usual as $O^T DO$ and then applying the transformation of O^T we get the quadric looks like

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = 0$$

Now if the given determinant was zero at least one of the eigenvalues are zero. WLOG let $\lambda_3 = 0$. This gives us

$$\lambda_1 X^2 + \lambda_2 Y^2 = 0$$

$$\implies (X - \sqrt{\frac{\lambda_2}{\lambda_1}} Y)(X + \sqrt{\frac{\lambda_2}{\lambda_1}} Y) = 0$$

Which shows the quadric is factorisable

Problem 3

Conversely assume the matrix is factorisable ie

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = (a_1 X + b_1 Y + c_1 Z)(a_2 X + b_2 Y + c_2 Z)$$

Now let us assume all eigenvalues are non zero. We know that the equation $a_1 X + b_1 Y + c_1 Z = 0$ has a non zero solution for X,Y,Z (use fundamental theorem).
If all eigenvalues are non zero, for this X,Y,Z we have LHS¿0 but RHS=0, contradiction.

Problem 4

Show that a (3x3) orthogonal matrix has eigenvalue $+1$ or -1. Further if determinant of matrix is 1 then 1 is necessarily an eigenvalue

Problem 4

Let $O \in R^{3\times 3}$ be orthogonal. Let $Ov = \lambda v$. Then

$$v^T O^T = \lambda v^T$$

Thus

$$v^T O^T O v = \lambda^2 v^T v$$

Thus $|\lambda| = 1$ Now the characteristic polynomial is a three degree polynomial with all coefficients real. Thus it must have at least one real root which means either 1 or -1 must be a root

Problem 4

Now suppose its given that determinant of A is one.
We know that the product of all eigenvalues is nothing but the determinant of the matrix. Since our characteristic polynomial is a three dgree one with real coefficients we get two possible cases
Case 1 : All roots real
Suppose none of the roots are 1, Then the product would be $(-1)^3 = (-1) \neq det(A) = 1$ Contradiction
Case 2 : One real root and two complex conjugate roots
Let the real root be α and complex roots be $\beta, \bar{\beta}$, we have

$$\alpha\beta\bar{\beta} = 1$$

$$\implies \alpha = 1$$

Hence Proved

Problem 5

Let A be a (3x3) orthogonal real matrix and V be a unit vector with eigenvalue 1 or -1. Let $\alpha + i\beta$ be a complex eigenvalue and $\delta + i\sigma$ be the corresponding eigenvector. Prove that

$$A\delta = \alpha\delta - \beta\sigma$$

$$A\sigma = \beta\delta + \alpha\sigma$$

Is it possible to arrange it so that $\{v, \delta, \sigma\}$ are orthonormal? Call $O = [v \quad \delta \quad \sigma]$, what is $O^T A O$

Problem 5

We have

$$A(\delta + i\sigma) = (\alpha + i\beta)(\delta + i\sigma)$$

Expand and compare real and imaginary parts to get

$$A\delta = \alpha\delta - \beta\sigma$$

$$A\sigma = \beta\delta + \alpha\sigma$$

Problem 5

By spectral theorem we know that,

$$\langle (\delta + i\sigma)\, v \rangle = 0$$

We conclude that

$$\langle \delta\, v \rangle = \langle \sigma\, v \rangle = 0$$

Now let

$$O = [v \quad \delta \quad \sigma]$$

Problem 5

We can observe

$$AO = [Av \quad A\delta \quad A\sigma]$$

$$AO = [v \quad \delta \quad \sigma] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}$$

Thus

$$O^T AO = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}$$