MA106 Linear Algebra 2022

Tutorial 4 D1-T4

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Problem 1

Suppose P is non-singular $(n \times n)$, A and B are both $(n \times n)$ matrices, show that A and $P^{-1}AP = B$ have same characteristic equation.

These are called similar matrices

Problem 1

This follows from the definition of characteristic equation.

(A vector which satisfies $Mv = xv$ for some number x is called an eigenvector of the matrix M and x is called the eigenvalue of M corresponding to v. (v is called an eigenvector corresponding to x.) The condition $Mv = xv$ can be rewritten as $(M - xI)v = 0$. This equation says that the matrix $(M - xI)$ takes v into the 0 vector, which implies that $(M - xI)$ cannot have an inverse so that its determinant must be 0. The equation $det(M - xI) = 0$ is a polynomial equation in the variable x for given M. It is called the characteristic equation of the matrix M. You can solve it to find the eigenvalues x, of M.)

$$\begin{aligned} p_B(x) &= det(xI - B) \\ &= det(xI - P^{-1}AP) \\ &= det(xP^{-1}P - P^{-1}AP) \\ &= det(P^{-1}(xI - A)P) \\ &= det(xI - A) = p_A(x) \end{aligned}$$

This is a very useful property of similar matrices.

Problem 2

Show that if A,B are square matrices of the same size $(n \times n)$
$I_n - (AB)$ is invertible iff $I_n - (BA)$ is
Do AB and BA have the same eigenvalues?

Problem 2

We will prove if $I_n - (AB)$ is invertible then $I_n - (BA)$ is also invertible. The reverse direction is similar
$I_n - (AB)$ is invertible $\implies$ there exists C such that

$$C(I - AB) = I$$

$$\implies C - CAB = I$$

$$\implies BCA - BCABA = BA$$

$$\implies BCA(I - BA) = BA$$

$$\implies (BCA + I)(I - BA) = I$$

Hence proved

Problem 3

Prove that if nullity of A is k then x^k divides characteristic polynomial det(x**I**-A)

Problem 3

Let us first analyse a property of the characteristic polynomial

Useful Fact

Given $A \in K^{n \times n}$,

$$p_A(x) = det(xI - A) = x^n + c_1 x^{n-1} \cdots + c_n$$

Then,

$$c_k = (-1)^k \times (\text{sum of } k \times k \text{ principal minors})$$

Does this seem useful?

Problem 3

We know, by the Rank-Nullity Theorem
$N(A) = k \implies R(A) = n - k$. Thus, using determinant rank, we know every submatrix of $size > n - k$ will have 0 determinant.
Hence, directly from the above fact, we see the coefficients of powers of x less than k is directly 0 and thus, we have estabilished divisibility.
QED

Problem 4

Find eigenvalues and eigenvectors of

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 7 & 2 & 1 \end{bmatrix}$$

Problem 4

Clearly, we can evaluate the value of the eigenvalues using the characteristic equation

$$p_A(\lambda) = (\lambda - 5)(\lambda + 1)^2 = 0 \implies \lambda = \{5, -1, -1\}$$

Now, we evaluate the null space of (A+I) and (A-5I) respectively to find a set of eigenvectors as

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Problem 5

Find eigenvalues and eigenvectors of

$$\begin{bmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

Problem 5

Clearly, we can evaluate the value of the eigenvalues using the characteristic equation

$$p_A(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \implies \lambda = \{1, 2, 3\}$$

Now, we evaluate the null space of (A-I), (A-2I) and (A-3I) respectively to find a set of eigenvectors as

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$