# MA106 Linear Algebra 2022

## Tutorial 3 D1-T4

Kartik Gokhale

Department of Computer Science & Engineering
IIT Bombay

2022

# Problem 1

Let A be a $2 \times n$ matrix of real numbers. Is there a relationship between the $det(AA^T)$ and sum of the $(2 \times 2)$ principal minors of $det(A^TA)$? Is there a similar such relationship for $3 \times n$ matrices?

# Problem 1

Let us brute force it and see what happens

$$A = \begin{bmatrix} u_1 & \cdots & u_n \\ v_1 & \dots & v_n \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \sum u_i v_i & \sum u_i^2 \\ \sum v_i^2 & \sum u_i v_i \end{bmatrix}$$

$$A^T A = \begin{bmatrix} u_i v_j + u_j v_i \end{bmatrix}$$

$$M_{kt} = \begin{bmatrix} 2u_k v_k & u_t v_k + v_t u_k \\ u_t v_k + v_t u_k & 2u_t v_t \end{bmatrix}$$

Let us evaluate the sum of determinants

# Problem 1

$$\sum_{k<t} det(M_{kt}) = \sum_{k<t} det\left(\begin{bmatrix} 2u_k v_k & u_t v_k + v_t u_k \\ u_t v_k + v_t u_k & 2u_t v_t \end{bmatrix}\right)$$
$$= \sum_{k<t} u_k v_k u_t v_t - u_t^2 v_t^2 - u_k^2 v_k^2$$
$$= \det(AA^T)$$

QED

A similar result holds not only for $3 \times n$ matrices but for all $k \times n, k \leq n$ matrices[1]

[1] Why not for $k > n$?

# Problem 2

Show that

$$det\left(\begin{bmatrix} \cos\alpha & 1 & 0 & \cdots & 0 \\ 1 & 2cos\alpha & 1 & \cdots & 0 \\ 0 & 1 & 2\cos\alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2\cos\alpha \end{bmatrix}\right) = \cos n\alpha$$

# Problem 2

We will use induction. Trivially, for n=2, we have

$$2\cos^2\alpha - 1 = \cos 2\alpha$$

By the Induction hypothesis

$$D_k = 2\cos\alpha\cos(k-1)\alpha - \cos(k-2)\alpha$$

Hence,

$$D_k = cosk\alpha$$

QED

# Problem 3

Suppose $\langle x, y \rangle$ is a Hermitian product. Show that

$$4\langle x, y \rangle = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$$

# Problem 3

Let us recall some properties about the Hermitian product.

$$\|x\|^2 = \langle x, x\rangle$$
$$\langle x, ay_1 + by_2\rangle = a\langle x, y_1\rangle + b\langle x, y_2\rangle$$
$$\langle ax_1 + bx_2, y\rangle = \bar{a}\langle x_1, y\rangle + \bar{b}\langle x_2, y\rangle$$

Trivial from here

$$\|x+y\|^2 - \|x-y\|^2 = 2\langle x, y\rangle + 2\langle y, x\rangle$$
$$i\|x+iy\|^2 - i\|x-iy\|^2 = 2\langle x, y\rangle - 2\langle y, x\rangle$$
$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

QED

# Problem 4

Sheet 4 Problem 4

# Problem 4

$$\{[1,1,0,0],\ \frac{1}{2}[1,-1,2,0],\ \frac{1}{3}[1,-1,-1,3],\ \frac{1}{2}[-1,1,1,1],\ \mathbf{0},\ \mathbf{0}\}.$$

The 4 nonzero elements give an orthogonal basis. $[-2,-1,1,2]$ must be in linear span.

To verify Bessel's equality, we first note that

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}[1,1,0,0],\ \mathbf{u}_2 = \frac{1}{\sqrt{6}}[1,-1,2,0],\ \mathbf{u}_3 = \frac{1}{2\sqrt{3}}[1,-1,-1,3],\ \mathbf{u}_4 = \frac{1}{2}[-1,1,1,1].$$

Now $\sum(\mathbf{v}\cdot\mathbf{u}_j)^2 = \frac{9}{2} + \frac{1}{6} + \frac{4}{3} + 4 = 10 = \|\mathbf{v}\|^2$. (Verified!)

# Problem 5

Sheet 4 Problem 8

# Problem 5

$$\begin{aligned}
\mathbf{w}_1 = \mathbf{v}_1 &= [1, \imath, 0, 0, 0] \\
\mathbf{w}_2 = [0, 1, \imath, 0, 0] - \frac{-\imath}{2}[1, \imath, 0, 0, 0] &= \frac{1}{2}[\imath, 1, 2\imath, 0, 0] \\
\text{Similarly, } \mathbf{w}_3 &= \frac{1}{3}[-1, \imath, 1, 3\imath, 0] \\
\text{and } \mathbf{w}_4 &= \frac{1}{4}[-\imath, -1, \imath, 1, 4\imath].
\end{aligned}$$

The corresponding orthonormal (unitary) set is

$$\left\{ \frac{1}{\sqrt{2}}[1, \imath, 0, 0, 0],\ \frac{1}{\sqrt{6}}[\imath, 1, 2\imath, 0, 0],\ \frac{1}{2\sqrt{3}}[-1, \imath, 1, 3\imath, 0],\ \frac{1}{2\sqrt{5}}[-\imath, -1, \imath, 1, 4\imath] \right\}.$$

$\|[1, \imath, 1, \imath, 1]\|^2 = 5$. OTOH,

$$\begin{aligned}
|\langle \mathbf{v}, \mathbf{u}_1 \rangle|^2 &= 2, & |\langle \mathbf{v}, \mathbf{u}_2 \rangle|^2 &= 2/3, \\
|\langle \mathbf{v}, \mathbf{u}_3 \rangle|^2 &= 4/3, & |\langle \mathbf{v}, \mathbf{u}_4 \rangle|^2 &= 4/5.
\end{aligned}$$

The sum is $24/5 < 5$, hence can not be in the linear span.

# Problem 6

Show that if A is $n \times n$ complex matrix whose rows are orthonormal, then so are its columns

# Problem 6

For the beauty of this concise proof, let me represent OC as orthonormal columns and OR as orthonormal rows

$$\text{OR} \implies (AA^{\dagger} = I) \implies (A^{\dagger} = A^{\dagger}AA^{\dagger}) \implies (A^{\dagger}A = I) \implies \text{OC}$$

# Problem 7

$$V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix}, W = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2i \\ -1 \end{bmatrix}$$

Find $U \in \mathcal{C}^3$ such that the 3 are orthonormal. Any other way than Gram Schmidt?

# Problem 7

A trivial method exists using Gram Schmidt Orthonormalisation. Consider the trivial basis of $\mathcal{C}^3$ and orthormalize them with respect to the given vectors. At least one is bound to give you U
Another method would be to use the results of Problem 6. Construct a vector U such that the augmented matrix formed has orthonormal rows.

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Can you think of any other method?