MA106 Linear Algebra 2022

Tutorial 2 D1-T4

Kartik Gokhale

Department of Computer Science & Engineering
IIT Bombay

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Problem 1

Suppose A,B are real (nxn) matrices such that A+iB is invertible
Show that

$$det\left(\begin{bmatrix} A & B \\ -B & A \end{bmatrix}\right) > 0$$

Problem 1

We know det(A+iB) and det(A-iB) $\neq 0$. Following some elementary row operations, we will obtain the necessary conditions.

Problem 2

Numbers 20604, 53227, 25755, 20927 and 78421 are all divisible by 17. Show that

$$\begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 7 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{bmatrix}$$

is divisible by 17

Problem 2

We need to find a clever elementary transformation. Observe that the numbers given are rows of the matrix So, we perform the following

$$C_1 = \sum_{i=1}^{5} 10^{i-1} C_{6-i}$$

which results in

$$\begin{bmatrix} 20604 & 0 & 6 & 0 & 4 \\ 53277 & 3 & 2 & 7 & 7 \\ 25755 & 5 & 7 & 5 & 5 \\ 20927 & 0 & 9 & 2 & 7 \\ 78421 & 8 & 4 & 2 & 1 \end{bmatrix}$$

Taking determinant of the matrix with respect to column 1, we obtain

$$det = 20604X + 53277Y + 25755Z + +20927A + 78421B$$

which is divisible by 17

Problem 3

Show that a necessary condition for $x^2 + ax + b$ and $x^2 + px + q$ to have a common root is

$$\begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & p & q & 0 \\ 0 & 1 & p & q \end{bmatrix} = 0$$

Problem 3

Assume a common root exists (as we want to find a necessary condition, not a sufficient one). Let this root be C.
Consider the following set of 4 equations

$$c^2 + ac + b = 0$$
$$c^2 + pc + q = 0$$
$$c^3 + ac^2 + bc = 0$$
$$c^3 + pc^2 + qc = 0$$

Consider this to be a system of equations in C. The necessary condition for C to exist is that determinant of coefficient matrix is non-zero.

$$\begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & p & q & 0 \\ 0 & 1 & p & q \end{bmatrix} = 0$$

Problem 4

Find the values of β for which Cramer's rule is applicable. For the remaining value(s) of β, find the number of solutions.

$$x + 2y + 3z = 20$$

$$x + 3y + z = 13$$

$$x + 6y + \beta z = \beta$$

Problem 4

For Cramers rule to be applicable all we need is that the determinant of the coefficients should be non zero. The coefficient matrix here is

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & \beta \end{bmatrix}$$

This has a determinant of $\beta + 5$.
Thus for $\beta = -5$ Cramers rule is not applicable.
In this case you can find the augmented matrix, perform gaussian elimination and observe that the system has no solutions (since rank of matrix is less than rank of augmented matrix).

Problem 5

Find whether the following set of vectors is linearly dependent or independent:

$$a\hat{i} + b\hat{j} + c\hat{k}$$
$$b\hat{i} + c\hat{j} + a\hat{k}$$
$$c\hat{i} + a\hat{j} + b\hat{k}$$

Problem 5

Consider the matrix of the column vectors and its determinant:

$$det\left(\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\right) = 3abc - a^3 - b^3 - c^3$$

$$3abc - a^3 - b^3 - c^3 = -\frac{1}{2}(a+b+c)((a-b)^2+(b-c)^2+(c-a)^2$$

which is 0 when $a = b = c$ or when $a + b + c = 0$.
This is the condition for linear dependence.

Problem 6

Invert the matrix

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Problem 6

We will evaluate minor and cofactor matrix respectively

$$M = \begin{bmatrix} 1/240 & 1/60 & 1/72 \\ 1/60 & 4/45 & 1/12 \\ 1/72 & 1/12 & 1/12 \end{bmatrix}, C = \begin{bmatrix} 1/240 & -1/60 & 1/72 \\ -1/60 & 4/45 & -1/12 \\ 1/72 & -1/12 & 1/12 \end{bmatrix}$$

$det(H) = \frac{1}{2160}$. Thus,

$$H^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

Problem 7

Show that if Wronskian is non-zero and $c_1 f_1 + \ldots + c_n f_n = 0$,then $c_1 = c_2 = \ldots = c_n = 0$

Problem 7

Simply differentiate the given expression n times.

$$c_1 f_1 + \ldots + c_n f_n = 0$$

We obtain

$$W \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$$

Thus, when det(W) is not 0, the c-matrix must be all zeros.$(Ax = 0, det(A) \neq 0 \implies x = 0)$