MA106 Linear Algebra 2022

Tutorial 1 D1-T4

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Problem 1

List all possibilities for the reduced row echelon matrices of order 4×4 having exactly k pivots $\forall k \in \{0, 1, 2, 3, 4\}$

Problem 1

For 0 pivots, the only possibility is the 0 matrix.
For 1 pivot, there are 4 possibilities

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with 3,2,1 and 0 degrees of freedom respectively.

Problem 1

For 3 pivots, there are 4 possibilities

$$\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with 3,2,1,0 degrees of freedom respectively. For 4 pivots, there is one possibility, the identity matrix with 0 degrees of freedom

Problem 1

For 3 pivots, there are 4 possibilities

$$\begin{bmatrix}1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}$$

with 3,2,1,0 degrees of freedom respectively. For 4 pivots, there is one possibility, the identity matrix with 0 degrees of freedom
Can you think of the number of possibilities for a $n \times n$ matrix with $k, k \leq n$ pivots?

Problem 2

Find whether the following set of vectors is linearly independent

1. $[1, -1, 1], [1, 1, -1], [-1, 1, 1], [0, 1, 0]$
2. $[1, 9, 9, 8], [2, 0, 0, 3], [2, 0, 0, 8]$

Problem 2

Clearly, the first set is not linearly independent

$$2[0,1,0] = [1,1,-1] + [-1,1,1]$$

Alternatively

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, not linearly independent(why?).
Also, observe that any 4-tuple of vectors in ‘3d’ space is linearly dependent.

Problem 2

For the second part

$$\begin{bmatrix} 1 & 2 & 2 \\ 9 & 0 & 0 \\ 9 & 0 & 0 \\ 8 & 3 & 8 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, there are 3 pivots thus the rank is 3 and the vectors are linearly independent.

Are any 3 vectors in '4d' linearly independent?

Problem 3

Find the ranks of the following matrices

1. $$\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$$
2. $$\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix}, m^2 \neq n^2$$
3. $$\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

Problem 3

Recall the determinant rank of a matrix (which equals the rank of a matrix)

Determinant Rank

Determinant rank of a matrix is k if there exists a $k \times k$ submatrix with non-zero determinant and every $(k+1) \times (k+1)$ matrix has zero determinant

Problem 3

- In the first subpart, all the 3 2×2 submatrices have zero determinant and there exists a scalar with non-zero value thus the rank is 1
- Considering the top 2×2 submatrix, since its determinant is non-zero, we have rank as 2 (What about the 3×3 determinants?)
- The bottom 3×3 determinant is non-zero and thus, rank is 3

Alternatively, solve the above using elementary row operations

Problem 4

For $a < b$, consider

$$\begin{aligned} x + y + z &= 1 \\ ax + by + 2z &= 3 \\ a^2x + b^2y + 4z &= 9 \end{aligned}$$

Find pairs (a,b) for which system has infinitely many solutions.

Problem 4

The necessary condition for infinite solutions is that det(A) = 0 where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & 2 \\ a^2 & b^2 & 4 \end{bmatrix}, det(A) = (b-a)(2-a)(2-b)$$

which has solutions only when a=2 or b=2. The above conditions also ensures existence of solution ensures infinitely many solutions.[1]

[1] Think on this

Problem 4

Case 1: a=2: Thus,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & 2 \\ 4 & b^2 & 4 \end{bmatrix}$$

Consider the augmented matrix with the values of LHS

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & b & 2 & 3 \\ 4 & b^2 & 4 & 9 \end{bmatrix}$$

The condition for having a solution is when rank of above is 2 which is ensured when b=3, which will also ensure infinite solutions as a=2. Thus (2,3) is a solution.

Case 2: b=2: No solution for (a,b) as we require a=3 and $a < b$

Problem 5

Show that the row space of a matrix does not change by row operations. Show that the dimension of the columns space is unchanged by column operations

Problem 5

The first part is easy to see as the new rows are linear combinations of the previous rows. Thus, the final rows are also linear combinations of the original rows (and vice-versa?[2])

[2] Not too trivial

Problem 5

Suppose we apply row transforms through matrix E to a matrix [C1,C2...Cn], the new columns obtained are [EC1,EC2...ECn] which are also linearly independent due to invertibility of E[3]

[3] Oops, again non-trivial

Problem 6

Consider the 4×7 system Ax=b

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 & -3 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \end{array}\right]$$

- Find rank(k) and nullity of A
- Is the system solvable?
- Find a $k \times k$ submatrix of A with non-zero determinant.
- Find a basis for null space of A.
- Find a basis for column space of A
- Find the complete set of solutions
- Which are the free variables?

Problem 6

Performing Elementary row operations

$$\begin{bmatrix}1 & 0 & 2 & -1 & 1 & -2 & -1\\ 2 & 2 & 6 & 0 & 4 & 2 & 4\\ 1 & -1 & 1 & -2 & 0 & -5 & -4\\ 2 & 2 & 6 & 0 & 4 & 2 & 4\end{bmatrix} \mapsto \begin{bmatrix}1 & 0 & 2 & -1 & 1 & -2 & -1\\ 0 & 2 & 2 & 2 & 2 & 6 & 6\\ 0 & -1 & -1 & -1 & -1 & -3 & -3\\ 0 & 2 & 2 & 2 & 2 & 6 & 6\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 2 & -1 & 1 & -2 & -1\\ 0 & 2 & 2 & 2 & 2 & 6 & 6\\ 0 & -1 & -1 & -1 & -1 & -3 & -3\\ 0 & 2 & 2 & 2 & 2 & 6 & 6\end{bmatrix} \mapsto \begin{bmatrix}1 & 0 & 2 & -1 & 1 & -2 & -1\\ 0 & 2 & 2 & 2 & 2 & 6 & 6\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Rank is 2 and nullity is 5

Problem 6

Augmented matrix becomes

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, basic consistency is satisfied. Hence, system is solvable. Top right 2×2 matrix will have non-zero determinant(How did we choose this?)

Problem 6

A general vector in the null space can be expressed as
$[-2k_3, +k_4 - k_5 + 2k_6 - k_7, -k_3 - k_4 - k_5 - 3k_6 - 3k_7, k_3, k_4, k_5, k_6, k_7]^T$
Thus, a basis for null space is

$$\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

And hence, a basis for column space is

$$[1, 2, 1, 2]^T, [0, 2, -1, 2]^T$$

Problem 6

Basic Solution is $B = [1, 4, 0, 0, 0, 0, 0]^T$ and thus, the complete solution set is

$$S = B + \sum_{i=1}^{5} k_i v_i, \, v_i \in \mathcal{N}(\mathcal{A}), k_i \in \mathcal{R}$$

Free Variables $F = \{x_3, x_4, x_5, x_6, x_7\}$