MA106 Linear Algebra 2022

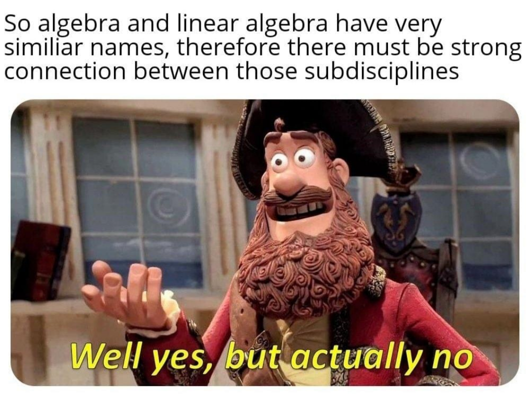
Tutorial 0 D1-T4

Kartik Gokhale

Department of Computer Science & Engineering
IIT Bombay

2022

Welcome to MA106: Linear Algebra.
The course serves as a background for a lot of courses in image/signal processing, data science and machine learning etc.
What the course is thought to be about:



What the course is actually about:

- Systems of linear equations and concepts of vector spaces
- linear transformations
- Eigenvalue problem
- Spectral theorem(s)
- Abstract vector spaces

Problem 1

Let $\hat{a}, \hat{b}$ be unit vectors in $\mathcal{R}^3$. Discuss whether the equation $\hat{a} \times \hat{x} = \hat{b}$ has solutions in $\mathcal{R}^3$; $\times$ is the cross product.

Problem 1

Speaking geometrically, we can note the following points about this problem

Obtaining Necessary Condition(s)

- Vectors $\hat{a}$ and 'x', should it exist, will 'span' a plane.
- Vector $\hat{b}$ must be perpendicular to this plane formed.
- Thus, $\hat{b}$ must be perpendicular to $\hat{a}$ independent of x.

Thus, we obtain our first necessary condition geometrically, stating that $\hat{a} \cdot \hat{b} = 0$.

Problem 1

Is this condition sufficient?
Yes! This is also a sufficient condition. How?
Consider the plane spanned by $\hat{a}$ and $\hat{b}$. Consider a perpendicular to this plane, obtained by

$$\hat{c} = \hat{b} \times \hat{a}$$

Now, consider

$$\begin{aligned} \hat{a} \times \hat{c} &= \hat{a} \times (\hat{b} \times \hat{a}) \\ &= \hat{b} \end{aligned}$$

Thus, $\hat{b} \times \hat{a}$ is an acceptable solution to the required equation.
Thus, we have the sufficient condition as well.

Problem 1

We have a solution uniquely specified by $\hat{b}$ and $\hat{a}$. Is this solution unique?
No. Clearly,

$$\hat{a} \times (\hat{b} \times \hat{a} + \lambda \hat{a}) = \hat{b} + \cancelto{0}{\lambda \hat{a} \times \hat{a}}$$
$$= \hat{b}$$

Problem 2

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $p = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{R}^2; (x_1^2 + x_2^2 = 1)$. What can you say about the set $\{p, Ap, A^2p, \ldots\}$? Is it a finite or infinite set?

Problem 2

Matrix A looks particularly tempting(try to visualise what this matrix does geometrically). Let us, be motivated by this and attempt to find a closed form expression for A^n.

Claim

Closed form expression for the n^{th} power of the matrix is

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

How to prove this?
Easy - Induction!

Problem 2

Let us now parametrize $\vec{p}$. We can do this as $p = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}, \phi \in \mathcal{R}$

Now, we can find a closed form expression for $A^n p$, which is rather trivially,

$$A^n p = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} = \begin{bmatrix} \cos(n\theta + \phi) \\ \sin(n\theta + \phi) \end{bmatrix}$$

Now, to consider the given set, if we have a finite number of elements, say m, then the $(m+1)^{st}$ element must be the same as the 1^{st} element and thus, $m\theta$ must be an integral multiple of 2π. Completing this argument, we can assess

- If θ is a rational multiple of π, the set is finite.
- If θ is not a rational multiple of π, the set is countably infinite[1].

[1] Why not uncountable

Problem 3

Consider the equation $x^2 + y^2 - z^2 + 7xy - 3yz + 6xz = 3$. Write it in the form $\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for some symmetric (3×3) matrix A. Is A unique? What if we drop the symmetry requirement?

Problem 3

Let us consider the very general expression

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Expanding and comparing coefficients, we obtain

$$\begin{aligned} a &= 1, & e &= 1, & i &= -1 \\ (b+d) &= 7, & (c+g) &= -3, & (f+h) &= 6 \end{aligned}$$

The solutions are clearly not unique. However, if we impose a restriction on A that is is symmetric, we obtain

$$b = d = 3.5, c = g = -1.5, f = h = 3$$

and thus, we obtain a unique symmetric A

$$A = \begin{bmatrix} 1 & 3.5 & 3 \\ 3.5 & 1 & -1.5 \\ 3 & -1.5 & -1 \end{bmatrix}$$

Problem 4

Recall the notion of an invertible matrix. How would you decide if a (3×3) matrix is invertible or not? If $\hat{u}$ is a unit vector in $\mathcal{R}^3$ - column vector. Is $I - uu^T$ invertible? Discuss the map $f : \mathcal{R}^3 \rightarrow \mathcal{R}^3; f(x) = (I - 2uu^T)x$ geometrically

Problem 4

Claim

A matrix can be invertible if we can ensure that[a]

$$Ax = \vec{0} \implies x = \vec{0}$$

[a] Equivalent to the condition imposable on the determinant

Proof: Suppose we had $Av_1 = Av_2$ then $v_1 = v_2$ and thus, for a given value of Ax, we have a unique value of x. Thus, in the range of Ax, the transformation represented by A is invertible

Problem 4

Consider

$$(I - uu^T)u = u - u = 0$$

Since $I - uu^T$ does NOT satisfy the above condition, it is not invertible.

Claim

The map $f : \mathcal{R}^3 \rightarrow \mathcal{R}^3; f(x) = (I - 2uu^T)x$ is a reflection of a given vector about the plane perpendicular to u.

Proof:

- Consider vectors perpendicular to $\hat{u}$, Clearly, $(I - 2uu^T)x = x$
- Consider vectors parallel to $\hat{u}$, Clearly, $(I - 2uu^T)x = -x$
- Thus, a general vector has its component along $\hat{u}$ reversed and component perpendicular to $\hat{u}$ unchanged

Problem 5

Find 2 mutually perpendicular unit vectors $\hat{u}, \hat{v}$ such that $\hat{u}, \hat{v}$ lie on the plane $x + y + z = 0$. Write out a parametrization for circle $x^2 + y^2 + z^2 = 1, x + y + z = 0$

Problem 5

This provides us with a required parametrization for the plane. We can express any point in the plane as a linear combination of the vectors obtained above.

$$\vec{x} = k_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, k_1, k_2 \in \mathcal{R}$$

$$(k_1, k_2) \mapsto (k_1 + k_2, k_1 - k_2, -2k_1)$$

We also have a parametrization for the unit circle as required. Thus, we have

$$(\theta) \mapsto (\cos\theta, \sin\theta)$$

Thus, the circle is parameterized as

$$(\theta) \mapsto (a\cos\theta + b\sin\theta, a\cos\theta - b\sin\theta, -2a\cos\theta)$$

$(a = \frac{1}{\sqrt{6}}, b = \frac{1}{\sqrt{2}})$

Problem 6

Utilising

$$||Ax|| = x^T A^T AX$$

We can conclude that

$$A^T A = I$$