MA106 Linear Algebra 2022

Tutorial 0 D1-T4

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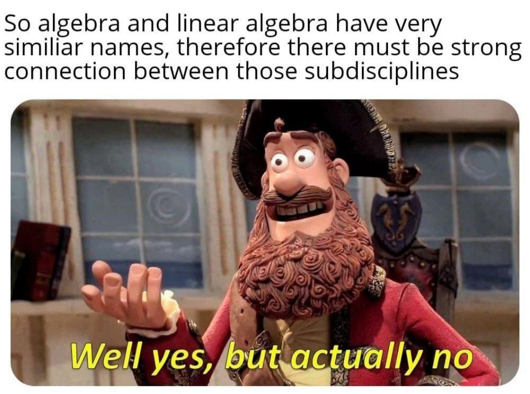
Introduction

Welcome to MA106: Linear Algebra.
The course serves as a background for a lot of courses in image/signal processing, data science and machine learning etc.
What the course is thought to be about:



Introduction

What the course is actually about:

- Systems of linear equations and concepts of vector spaces
- linear transformations
- Eigenvalue problem
- Spectral theorem(s)
- Abstract vector spaces

Problem 1

Let $\hat{a}, \hat{b}$ be unit vectors in $\mathcal{R}^3$. Discuss whether the equation $\hat{a} \times \hat{x} = \hat{b}$ has solutions in $\mathcal{R}^3$; $\times$ is the cross product.

Problem 1

Speaking geometrically, we can note the following points about this problem

Obtaining Necessary Condition(s)

- Vectors $\hat{a}$ and 'x', should it exist, will 'span' a plane.
- Vector $\hat{b}$ must be perpendicular to this plane formed.
- Thus, $\hat{b}$ must be perpendicular to $\hat{a}$ independent of x.

Thus, we obtain our first necessary condition geometrically, stating that $\hat{a} \cdot \hat{b} = 0$.

Problem 1

Is this condition sufficient?
Yes! This is also a sufficient condition. How?
Consider the plane spanned by $\hat{a}$ and $\hat{b}$. Consider a perpendicular to this plane, obtained by

$$\hat{c} = \hat{b} \times \hat{a}$$

Now, consider

$$\begin{aligned} \hat{a} \times \hat{c} &= \hat{a} \times (\hat{b} \times \hat{a}) \\ &= \hat{b} \end{aligned}$$

Thus, $\hat{b} \times \hat{a}$ is an acceptable solution to the required equation. Thus, we have the sufficient condition as well.

Problem 1

We have a solution uniquely specified by $\hat{b}$ and $\hat{a}$. Is this solution unique?
No. Clearly,

$$\hat{a} \times (\hat{b} \times \hat{a} + \lambda \hat{a}) = \hat{b} + \cancel{\lambda \hat{a} \times \hat{a}}^{\,0}$$
$$= \hat{b}$$

Problem 2

Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $p = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathcal{R}^2; (x_1^2 + x_2^2 = 1)$. What can you say about the set $\{p, Ap, A^2p, \ldots\}$? Is it a finite or infinite set?

Problem 2

Matrix A looks particularly tempting(try to visualise what this matrix does geometrically). Let us, be motivated by this and attempt to find a closed form expression for A^n.

Claim

Closed form expression for the n^{th} power of the matrix is

$$A^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

How to prove this?
Easy - Induction!

Problem 2

Let us now parametrize $\vec{p}$. We can do this as $p = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}, \phi \in \mathcal{R}$

Now, we can find a closed form expression for $A^n p$, which is rather trivially,

$$A^n p = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix} = \begin{bmatrix} \cos(n\theta + \phi) \\ \sin(n\theta + \phi) \end{bmatrix}$$

Now, to consider the given set, if we have a finite number of elements, say m, then the $(m+1)^{st}$ element must be the same as the 1^{st} element and thus, $m\theta$ must be an integral multiple of 2π. Completing this argument, we can assess

- If θ is a rational multiple of π, the set is finite.
- If θ is not a rational multiple of π, the set is countably infinite[1].

[1] Why not uncountable

Problem 3

Consider the equation $x^2 + y^2 - z^2 + 7xy - 3yz + 6xz = 3$. Write it in the form $\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for some symmetric (3×3) matrix A. Is A unique? What if we drop the symmetry requirement?

Problem 3

Let us consider the very general expression

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Expanding and comparing coefficients, we obtain

$$\begin{aligned} a &= 1, & e &= 1, & i &= -1 \\ (b+d) &= 7, & (c+g) &= -3, & (f+h) &= 6 \end{aligned}$$

The solutions are clearly not unique. However, if we impose a restriction on A that is is symmetric, we obtain

$$b = d = 3.5, c = g = -1.5, f = h = 3$$

and thus, we obtain a unique symmetric A

$$A = \begin{bmatrix} 1 & 3.5 & 3 \\ 3.5 & 1 & -1.5 \\ 3 & -1.5 & -1 \end{bmatrix}$$

Problem 4

Recall the notion of an invertible matrix. How would you decide if a (3×3) matrix is invertible or not? If $\hat{u}$ is a unit vector in $\mathcal{R}^3$ - column vector. Is $I - uu^T$ invertible? Discuss the map $f : \mathcal{R}^3 \to \mathcal{R}^3; f(x) = (I - 2uu^T)x$ geometrically

Problem 4

Claim

A matrix can be invertible if we can ensure that[a]

$$Ax = \vec{0} \implies x = \vec{0}$$

[a] Equivalent to the condition imposable on the determinant

Proof: Suppose we had $Av_1 = Av_2$ then $v_1 = v_2$ and thus, for a given value of Ax, we have a unique value of x. Thus, in the range of Ax, the transformation represented by A is invertible

Problem 4

Consider

$$(I - uu^T)u = u - u = 0$$

Since $I - uu^T$ does NOT satisfy the above condition, it is not invertible.

Claim

The map $f : \mathcal{R}^3 \rightarrow \mathcal{R}^3; f(x) = (I - 2uu^T)x$ is a reflection of a given vector about the plane perpendicular to u.

Proof:

- Consider vectors perpendicular to $\hat{u}$, Clearly, $(I - 2uu^T)x = x$
- Consider vectors parallel to $\hat{u}$, Clearly, $(I - 2uu^T)x = -x$
- Thus, a general vector has its component along $\hat{u}$ reversed and component perpendicular to $\hat{u}$ unchanged

Problem 5

Find 2 mutually perpendicular unit vectors $\hat{u}, \hat{v}$ such that $\hat{u}, \hat{v}$ lie on the plane $x + y + z = 0$. Write out a parametrization for circle $x^2 + y^2 + z^2 = 1, x + y + z = 0$

Problem 5

Consider a vector in the given plane represented as $x + y + z = 0$, say

$$\vec{a} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Now, we consider the normal to the plane as

$$\vec{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Taking their cross product, we obtain

$$\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Dividing both vectors by their 2-norm, we obtain a required pair of unit vectors.

This provides us with a required parametrization for the plane. We can express any point in the plane as a linear combination of the vectors obtained above.

$$\vec{x} = k_1 \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, k_1, k_2 \in \mathcal{R}$$

$$(k_1, k_2) \mapsto (k_1 + k_2, k_1 - k_2, -2k_1)$$

We also have a parametrization for the unit circle as required. Thus, we have

$$(\theta) \mapsto (\cos\theta, \sin\theta)$$

Thus, the circle is parameterized as

$$(\theta) \mapsto (a\cos\theta + b\sin\theta, a\cos\theta - b\sin\theta, -2a\cos\theta)$$

$(a = \frac{1}{\sqrt{6}}, b = \frac{1}{\sqrt{2}})$

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Tutorial 1 D1-T4

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Problem 1

List all possibilities for the reduced row echelon matrices of order 4×4 having exactly k pivots $\forall k \in \{0, 1, 2, 3, 4\}$

Problem 1

For 0 pivots, the only possibility is the 0 matrix.
For 1 pivot, there are 4 possibilities

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with 3,2,1 and 0 degrees of freedom respectively.

Problem 1

For 3 pivots, there are 4 possibilities

$$\begin{bmatrix}1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}$$

with 3,2,1,0 degrees of freedom respectively. For 4 pivots, there is one possibility, the identity matrix with 0 degrees of freedom

Problem 1

For 3 pivots, there are 4 possibilities

$$\begin{bmatrix}1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & * & 0 \\ 0 & 1 & * & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0\end{bmatrix}$$

with 3,2,1,0 degrees of freedom respectively. For 4 pivots, there is one possibility, the identity matrix with 0 degrees of freedom
Can you think of the number of possibilities for a $n \times n$ matrix with $k, k \leq n$ pivots?

Problem 2

Find whether the following set of vectors is linearly independent

1. $[1, -1, 1], [1, 1, -1], [-1, 1, 1], [0, 1, 0]$
2. $[1, 9, 9, 8], [2, 0, 0, 3], [2, 0, 0, 8]$

Problem 2

Clearly, the first set is not linearly independent

$$2[0,1,0] = [1,1,-1] + [-1,1,1]$$

Alternatively

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, not linearly independent(why?).
Also, observe that any 4-tuple of vectors in '3d' space is linearly dependent.

Problem 2

For the second part

$$\begin{bmatrix} 1 & 2 & 2 \\ 9 & 0 & 0 \\ 9 & 0 & 0 \\ 8 & 3 & 8 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, there are 3 pivots thus the rank is 3 and the vectors are linearly independent.
Are any 3 vectors in ‘4d’ linearly independent?

Problem 3

Find the ranks of the following matrices

1. $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$
2. $\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix}, m^2 \neq n^2$
3. $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$

Problem 3

Recall the determinant rank of a matrix (which equals the rank of a matrix)

Determinant Rank

Determinant rank of a matrix is k if there exists a $k \times k$ submatrix with non-zero determinant and every $(k+1) \times (k+1)$ matrix has zero determinant

Problem 3

- In the first subpart, all the 3 2×2 submatrices have zero determinant and there exists a scalar with non-zero value thus the rank is 1
- Considering the top 2×2 submatrix, since its determinant is non-zero, we have rank as 2 (What about the 3×3 determinants?)
- The bottom 3×3 determinant is non-zero and thus, rank is 3

Alternatively, solve the above using elementary row operations

Problem 4

For $a < b$, consider

$$\begin{aligned} x + y + z &= 1 \\ ax + by + 2z &= 3 \\ a^2x + b^2y + 4z &= 9 \end{aligned}$$

Find pairs (a,b) for which system has infinitely many solutions.

Problem 4

The necessary condition for infinite solutions is that det(A) = 0 where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & 2 \\ a^2 & b^2 & 4 \end{bmatrix}, det(A) = (b-a)(2-a)(2-b)$$

which has solutions only when a=2 or b=2. The above conditions also ensures existence of solution ensures infinitely many solutions.[1]

[1] Think on this

Problem 4

Case 1: a=2: Thus,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & b & 2 \\ 4 & b^2 & 4 \end{bmatrix}$$

Consider the augmented matrix with the values of LHS

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & b & 2 & 3 \\ 4 & b^2 & 4 & 9 \end{bmatrix}$$

The condition for having a solution is when rank of above is 2 which is ensured when b=3, which will also ensure infinite solutions as a=2. Thus (2,3) is a solution.
Case 2: b=2: No solution for (a,b) as we require a=3 and $a < b$

Problem 5

Show that the row space of a matrix does not change by row operations. Show that the dimension of the columns space is unchanged by column operations

Problem 5

The first part is easy to see as the new rows are linear combinations of the previous rows. Thus, the final rows are also linear combinations of the original rows (and vice-versa?[2])

[2]Not too trivial

Problem 5

Suppose we apply row transforms through matrix E to a matrix [C1,C2...Cn], the new columns obtained are [EC1,EC2...ECn] which are also linearly independent due to invertibility of E[3]

[3] Oops, again non-trivial

Problem 6

Consider the 4×7 system Ax=b

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 & -3 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \end{array}\right]$$

- Find rank(k) and nullity of A
- Is the system solvable?
- Find a $k \times k$ submatrix of A with non-zero determinant.
- Find a basis for null space of A.
- Find a basis for column space of A
- Find the complete set of solutions
- Which are the free variables?

Problem 6

Performing Elementary row operations

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 1 & -2 & -1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 2 & -1 & 1 & -2 & -1 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 \\ 0 & -1 & -1 & -1 & -1 & -3 & -3 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 1 & -2 & -1 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 \\ 0 & -1 & -1 & -1 & -1 & -3 & -3 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 2 & -1 & 1 & -2 & -1 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank is 2 and nullity is 5

Problem 6

Augmented matrix becomes

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 0 & 2 & 2 & 2 & 2 & 6 & 6 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, basic consistency is satisfied. Hence, system is solvable.
Top right 2×2 matrix will have non-zero determinant(How did we choose this?)

Problem 6

A general vector in the null space can be expressed as
$[-2k_3, +k_4 - k_5 + 2k_6 - k_7, -k_3 - k_4 - k_5 - 3k_6 - 3k_7, k_3, k_4, k_5, k_6, k_7]^T$
Thus, a basis for null space is

$$\begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

And hence, a basis for column space is

$$[1, 2, 1, 2]^T, [0, 2, -1, 2]^T$$

Problem 6

Basic Solution is $B = [1, 4, 0, 0, 0, 0, 0]^T$ and thus, the complete solution set is

$$S = B + \sum_{i=1}^{5} k_i v_i, v_i \in \mathcal{N}(\mathcal{A}), k_i \in \mathcal{R}$$

Free Variables $F = \{x_3, x_4, x_5, x_6, x_7\}$

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Tutorial 2 D1-T4

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Problem 1

Suppose A,B are real (nxn) matrices such that A+iB is invertible
Show that

$$det\left(\begin{bmatrix} A & B \\ -B & A \end{bmatrix}\right) > 0$$

Problem 1

We know det(A+iB) and det(A-iB) $\neq 0$. Following some elementary row operations, we will obtain the necessary conditions.

Problem 2

Numbers 20604, 53227, 25755, 20927 and 78421 are all divisible by 17. Show that

$$\begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 7 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{bmatrix}$$

is divisible by 17

Problem 2

We need to find a clever elementary transformation. Observe that the numbers given are rows of the matrix So, we perform the following

$$C_1 = \sum_{i=1}^{5} 10^{i-1} C_{6-i}$$

which results in

$$\begin{bmatrix} 20604 & 0 & 6 & 0 & 4 \\ 53277 & 3 & 2 & 7 & 7 \\ 25755 & 5 & 7 & 5 & 5 \\ 20927 & 0 & 9 & 2 & 7 \\ 78421 & 8 & 4 & 2 & 1 \end{bmatrix}$$

Taking determinant of the matrix with respect to column 1, we obtain

$$det = 20604X + 53277Y + 25755Z + +20927A + 78421B$$

which is divisible by 17

Problem 3

Show that a necessary condition for $x^2 + ax + b$ and $x^2 + px + q$ to have a common root is

$$\begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & p & q & 0 \\ 0 & 1 & p & q \end{bmatrix} = 0$$

Problem 3

Assume a common root exists (as we want to find a necessary condition, not a sufficient one). Let this root be C.
Consider the following set of 4 equations

$$
\begin{aligned}
c^2 + ac + b &= 0 \\
c^2 + pc + q &= 0 \\
c^3 + ac^2 + bc &= 0 \\
c^3 + pc^2 + qc &= 0
\end{aligned}
$$

Consider this to be a system of equations in C. The necessary condition for C to exist is that determinant of coefficient matrix is non-zero.

$$
\begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & p & q & 0 \\ 0 & 1 & p & q \end{bmatrix} = 0
$$

Problem 4

Find the values of β for which Cramer's rule is applicable. For the remaining value(s) of β, find the number of solutions.

$$x + 2y + 3z = 20$$

$$x + 3y + z = 13$$

$$x + 6y + \beta z = \beta$$

Problem 4

For Cramers rule to be applicable all we need is that the determinant of the coefficients should be non zero. The coefficient matrix here is

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 6 & \beta \end{bmatrix}$$

This has a determinant of $\beta + 5$.
Thus for $\beta = -5$ Cramers rule is not applicable.
In this case you can find the augmented matrix, perform gaussian elimination and observe that the system has no solutions (since rank of matrix is less than rank of augmented matrix).

Problem 5

Find whether the following set of vectors is linearly dependent or independent:

$$a\hat{i} + b\hat{j} + c\hat{k}$$

$$b\hat{i} + c\hat{j} + a\hat{k}$$

$$c\hat{i} + a\hat{j} + b\hat{k}$$

Problem 5

Consider the matrix of the column vectors and its determinant:

$$det\left(\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}\right) = 3abc - a^3 - b^3 - c^3$$

$$3abc - a^3 - b^3 - c^3 = -\frac{1}{2}(a+b+c)((a-b)^2 + (b-c)^2 + (c-a)^2$$

which is 0 when $a = b = c$ or when $a + b + c = 0$.
This is the condition for linear dependence.

Problem 6

Invert the matrix

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Problem 6

We will evaluate minor and cofactor matrix respectively

$$M = \begin{bmatrix} 1/240 & 1/60 & 1/72 \\ 1/60 & 4/45 & 1/12 \\ 1/72 & 1/12 & 1/12 \end{bmatrix}, C = \begin{bmatrix} 1/240 & -1/60 & 1/72 \\ -1/60 & 4/45 & -1/12 \\ 1/72 & -1/12 & 1/12 \end{bmatrix}$$

$det(H) = \frac{1}{2160}$. Thus,

$$H^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}$$

Problem 7

Show that if Wronskian is non-zero and $c_1 f_1 + \ldots + c_n f_n = 0$,then $c_1 = c_2 = \ldots = c_n = 0$

Problem 7

Simply differentiate the given expression n times.

$$c_1 f_1 + \ldots + c_n f_n = 0$$

We obtain

$$W \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = 0$$

Thus, when det(W) is not 0, the c-matrix must be all zeros.$(Ax = 0, det(A) \neq 0 \implies x = 0)$

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Tutorial 3 D1-T4

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Problem 1

Let A be a $2 \times n$ matrix of real numbers. Is there a relationship between the $det(AA^T)$ and sum of the (2×2) principal minors of $det(A^T A)$? Is there a similar such relationship for $3 \times n$ matrices?

Problem 1

Let us brute force it and see what happens

$$A = \begin{bmatrix} u_1 & \cdots & u_n \\ v_1 & \dots & v_n \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \sum u_i v_i & \sum u_i^2 \\ \sum v_i^2 & \sum u_i v_i \end{bmatrix}$$

$$A^T A = \begin{bmatrix} u_i v_j + u_j v_i \end{bmatrix}$$

$$M_{kt} = \begin{bmatrix} 2u_k v_k & u_t v_k + v_t u_k \\ u_t v_k + v_t u_k & 2u_t v_t \end{bmatrix}$$

Let us evaluate the sum of determinants

Problem 1

$$\sum_{k<t} det(M_{kt}) = \sum_{k<t} det\left(\begin{bmatrix} 2u_k v_k & u_t v_k + v_t u_k \\ u_t v_k + v_t u_k & 2u_t v_t \end{bmatrix}\right)$$
$$= \sum_{k<t} u_k v_k u_t v_t - u_t^2 v_t^2 - u_k^2 v_k^2$$
$$= \det(AA^T)$$

QED

A similar result holds not only for $3 \times n$ matrices but for all $k \times n, k \leq n$ matrices[1]

[1] Why not for $k > n$?

Problem 2

Show that

$$det\left(\begin{bmatrix} \cos\alpha & 1 & 0 & \cdots & 0 \\ 1 & 2cos\alpha & 1 & \cdots & 0 \\ 0 & 1 & 2\cos\alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 2\cos\alpha \end{bmatrix}\right) = \cos n\alpha$$

Problem 2

We will use induction. Trivially, for n=2, we have

$$2\cos^2\alpha - 1 = \cos 2\alpha$$

By the Induction hypothesis

$$D_k = 2\cos\alpha\cos(k-1)\alpha - \cos(k-2)\alpha$$

Hence,

$$D_k = cos k\alpha$$

QED

Problem 3

Suppose $\langle x, y\rangle$ is a Hermitian product. Show that

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

Problem 3

Let us recall some properties about the Hermitian product.

$$\|x\|^2 = \langle x, x\rangle$$
$$\langle x, ay_1 + by_2\rangle = a\langle x, y_1\rangle + b\langle x, y_2\rangle$$
$$\langle ax_1 + bx_2, y\rangle = \bar{a}\langle x_1, y\rangle + \bar{b}\langle x_2, y\rangle$$

Trivial from here

$$\|x+y\|^2 - \|x-y\|^2 = 2\langle x, y\rangle + 2\langle y, x\rangle$$
$$i\|x+iy\|^2 - i\|x-iy\|^2 = 2\langle x, y\rangle - 2\langle y, x\rangle$$
$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

QED

Problem 4

Sheet 4 Problem 4

$$\{[1,1,0,0],\ \frac{1}{2}[1,-1,2,0],\ \frac{1}{3}[1,-1,-1,3],\ \frac{1}{2}[-1,1,1,1],\ \mathbf{0},\ \mathbf{0}\}.$$

The 4 nonzero elements give an orthogonal basis. $[-2,-1,1,2]$ must be in linear span.

To verify Bessel's equality, we first note that

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}[1,1,0,0],\ \mathbf{u}_2 = \frac{1}{\sqrt{6}}[1,-1,2,0],\ \mathbf{u}_3 = \frac{1}{2\sqrt{3}}[1,-1,-1,3],\ \mathbf{u}_4 = \frac{1}{2}[-1,1,1,1].$$

Now $\sum(\mathbf{v}\cdot\mathbf{u}_j)^2 = \frac{9}{2} + \frac{1}{6} + \frac{4}{3} + 4 = 10 = \|\mathbf{v}\|^2$. (Verified!)

Problem 5

Sheet 4 Problem 8

$$\begin{aligned} \mathbf{w}_1 = \mathbf{v}_1 &= [1, \imath, 0, 0, 0] \\ \mathbf{w}_2 = [0,1,\imath,0,0] - \frac{-\imath}{2}[1,\imath,0,0,0] &= \frac{1}{2}[\imath, 1, 2\imath, 0, 0] \\ \text{Similarly, } \mathbf{w}_3 &= \frac{1}{3}[-1, \imath, 1, 3\imath, 0] \\ \text{and } \mathbf{w}_4 &= \frac{1}{4}[-\imath, -1, \imath, 1, 4\imath]. \end{aligned}$$

The corresponding orthonormal (unitary) set is

$$\left\{ \frac{1}{\sqrt{2}}[1,\imath,0,0,0],\ \frac{1}{\sqrt{6}}[\imath,1,2\imath,0,0],\ \frac{1}{2\sqrt{3}}[-1,\imath,1,3\imath,0],\ \frac{1}{2\sqrt{5}}[-\imath,-1,\imath,1,4\imath] \right\}.$$

$\|[1,\imath,1,\imath,1]\|^2 = 5$. OTOH,

$$\begin{aligned} |\langle \mathbf{v}, \mathbf{u}_1 \rangle|^2 &= 2, & |\langle \mathbf{v}, \mathbf{u}_2 \rangle|^2 &= 2/3, \\ |\langle \mathbf{v}, \mathbf{u}_3 \rangle|^2 &= 4/3, & |\langle \mathbf{v}, \mathbf{u}_4 \rangle|^2 &= 4/5. \end{aligned}$$

The sum is $24/5 < 5$, hence can not be in the linear span.

Problem 6

Show that if A is $n \times n$ complex matrix whose rows are orthonormal, then so are its columns

Problem 6

For the beauty of this concise proof, let me represent OC as orthonormal columns and OR as orthonormal rows

$$\text{OR} \implies (AA^\dagger = I) \implies (A^\dagger = A^\dagger A A^\dagger) \implies (A^\dagger A = I) \implies \text{OC}$$

Problem 7

$$V = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix}, W = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2i \\ -1 \end{bmatrix}$$

Find $U \in \mathcal{C}^3$ such that the 3 are orthonormal. Any other way than Gram Schmidt?

Problem 7

A trivial method exists using Gram Schmidt Orthonormalisation. Consider the trivial basis of $\mathcal{C}^3$ and orthormalize them with respect to the given vectors. At least one is bound to give you U
Another method would be to use the results of Problem 6. Construct a vector U such that the augmented matrix formed has orthonormal rows.

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Can you think of any other method?

MA106 Linear Algebra 2022

Tutorial 4 D1-T4

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2022

Problem 1

Suppose P is non-singular $(n \times n)$, A and B are both $(n \times n)$ matrices, show that A and $P^{-1}AP = B$ have same characteristic equation.
These are called similar matrices

Problem 1

This follows from the definition of characteristic equation.

(A vector which satisfies $Mv = xv$ for some number x is called an eigenvector of the matrix M and x is called the eigenvalue of M corresponding to v. (v is called an eigenvector corresponding to x.) The condition $Mv = xv$ can be rewritten as $(M - xI)v = 0$. This equation says that the matrix $(M - xI)$ takes v into the 0 vector, which implies that $(M - xI)$ cannot have an inverse so that its determinant must be 0. The equation $det(M - xI) = 0$ is a polynomial equation in the variable x for given M. It is called the characteristic equation of the matrix M. You can solve it to find the eigenvalues x, of M.)

$$\begin{aligned} p_B(x) &= det(xI - B) \\ &= det(xI - P^{-1}AP) \\ &= det(xP^{-1}P - P^{-1}AP) \\ &= det(P^{-1}(xI - A)P) \\ &= det(xI - A) = p_A(x) \end{aligned}$$

This is a very useful property of similar matrices.

Problem 2

Show that if A,B are square matrices of the same size $(n \times n)$
$I_n - (AB)$ is invertible iff $I_n - (BA)$ is
Do AB and BA have the same eigenvalues?

Problem 2

We will prove if $I_n - (AB)$ is invertible then $I_n - (BA)$ is also invertible. The reverse direction is similar
$I_n - (AB)$ is invertible $\implies$ there exists C such that

$$C(I - AB) = I$$

$$\implies C - CAB = I$$

$$\implies BCA - BCABA = BA$$

$$\implies BCA(I - BA) = BA$$

$$\implies (BCA + I)(I - BA) = I$$

Hence proved

Problem 3

Prove that if nullity of A is k then x^k divides characteristic polynomial det(xI-A)

Problem 3

Let us first analyse a property of the characteristic polynomial

Useful Fact

Given $A \in K^{n \times n}$,

$$p_A(x) = det(xI - A) = x^n + c_1 x^{n-1} \cdots + c_n$$

Then,

$$c_k = (-1)^k \times (\text{sum of } k \times k \text{ principal minors})$$

Does this seem useful?

Problem 3

We know, by the Rank-Nullity Theorem
$N(A) = k \implies R(A) = n - k$. Thus, using determinant rank, we know every submatrix of $size > n - k$ will have 0 determinant.
Hence, directly from the above fact, we see the coefficients of powers of x less than k is directly 0 and thus, we have estabilished divisibility.
QED

Problem 4

Find eigenvalues and eigenvectors of

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 7 & 2 & 1 \end{bmatrix}$$

Problem 4

Clearly, we can evaluate the value of the eigenvalues using the characteristic equation

$$p_A(\lambda) = (\lambda - 5)(\lambda + 1)^2 = 0 \implies \lambda = \{5, -1, -1\}$$

Now, we evaluate the null space of (A+I) and (A-5I) respectively to find a set of eigenvectors as

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Problem 5

Find eigenvalues and eigenvectors of

$$\begin{bmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

Problem 5

Clearly, we can evaluate the value of the eigenvalues using the characteristic equation

$$p_A(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0 \implies \lambda = \{1, 2, 3\}$$

Now, we evaluate the null space of (A-I), (A-2I) and (A-3I) respectively to find a set of eigenvectors as

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

MA106 Linear Algebra 2022

Tutorial 5 D1-T4

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Problem 1

Identify the quadratic form and in at least one instance find the directions of the principal axes

- $2xy + 2yz + 2zx = 1$
- $x^2 - 2y^2 + 4z^2 + 6yz = 1$
- $-x^2 - y^2 + 2z^2 + 8xy_4xz + 4yz = 1$

Problem 1(i)

We have the quadratic form as

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

We choose A to be symmetric

Problem 1(i)

Thus we have

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We get eigenvalues as -1,-1,2
Thus this is a hyperboloid of 2 sheets

Problem 1(i)

For $\lambda = -1$ eigenvectors are

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 2$ eigenvectors are

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

To get the principal axes all is left to make this an orthogonal set, for this just apply Gram Schmidt procedure on the eigenvectors of $\lambda = -1$ (note different eigenvalues will already have orthogonal eigenvectors)

Problem 1(ii)

Thus we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 3 \\ 0 & 3 & 4 \end{bmatrix}$$

We get eigenvalues as $1, 1 \pm 3\sqrt{2}$
Thus this is a hyperboloid of 1 sheets

Thus we have

$$A = \begin{bmatrix} -1 & 4 & -2 \\ 4 & -1 & 2 \\ -2 & 2 & 2 \end{bmatrix}$$

We get eienvalues as -6,3,3

Thus this is a hyperboloid of 1 sheets

Problem 2

Compute integral

$$\iiint exp(-(2x^2 + 5y^2 + 2z^2 - 4xy - 2xz + 4yz))\, dx\, dy\, dz$$

Problem 2

Thus we have

$$A = \begin{bmatrix} 2 & -2 & -1 \\ -2 & 5 & 2 \\ -1 & 2 & 2 \end{bmatrix}$$

We get eiegenvalues as 1,1,7
We can thus write A as

$$A = ODO^T$$

where D=diag(1,1,7) and O is orthogonal

Problem 2

In effect we perform the transformation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow O^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The Jacobian for this is nothing but O^T which has a determinant of 1 or -1 which gives us $|det(J)| = 1$

Problem 2

Hence we are left with

$$\iiint exp(-(X^2 + Y^2 + 7Z^2))\, dX\, dY\, dZ$$

which can easily be computed

Problem 3

Show that $ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz$ factorizes into a product of linear factors (possibly with complex coefficients) iff

$$\left| \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \right| = 0$$

Problem 3

We can write A as usual as $O^T DO$ and then applying the transformation of O^T we get the quadric looks like

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = 0$$

Now if the given determinant was zero at least one of the eigenvalues are zero. WLOG let $\lambda_3 = 0$. This gives us

$$\lambda_1 X^2 + \lambda_2 Y^2 = 0$$

$$\implies (X - \sqrt{\frac{\lambda_2}{\lambda_1}} Y)(X + \sqrt{\frac{\lambda_2}{\lambda_1}} Y) = 0$$

Which shows the quadric is factorisable

Problem 3

Conversely assume the matrix is factorisable ie

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = (a_1 X + b_1 Y + c_1 Z)(a_2 X + b_2 Y + c_2 Z)$$

Now let us assume all eigenvalues are non zero. We know that the equation $a_1 X + b_1 Y + c_1 Z = 0$ has a non zero solution for X,Y,Z (use fundamental theorem).
If all eigenvalues are non zero, for this X,Y,Z we have LHS¿0 but RHS=0, contradiction.

Problem 4

Show that a (3x3) orthogonal matrix has eigenvalue $+1$ or -1. Further if determinant of matrix is 1 then 1 is necessarily an eigenvalue

Problem 4

Let $O \in R^{3\times 3}$ be orthogonal. Let $Ov = \lambda v$. Then

$$v^T O^T = \lambda v^T$$

Thus

$$v^T O^T O v = \lambda^2 v^T v$$

Thus $|\lambda| = 1$ Now the characteristic polynomial is a three degree polynomial with all coefficients real. Thus it must have at least one real root which means either 1 or -1 must be a root

Problem 4

Now suppose its given that determinant of A is one.
We know that the product of all eigenvalues is nothing but the determinant of the matrix. Since our characteristic polynomial is a three dgree one with real coefficients we get two possible cases
Case 1 : All roots real
Suppose none of the roots are 1, Then the product would be $(-1)^3 = (-1) \neq det(A) = 1$ Contradiction
Case 2 : One real root and two complex conjugate roots
Let the real root be α and complex roots be $\beta, \bar{\beta}$, we have

$$\alpha\beta\bar{\beta} = 1$$

$$\implies \alpha = 1$$

Hence Proved

Problem 5

Let A be a (3x3) orthogonal real matrix and V be a unit vector with eigenvalue 1 or -1. Let $\alpha + i\beta$ be a complex eigenvalue and $\delta + i\sigma$ be the corresponding eigenvector. Prove that

$$A\delta = \alpha\delta - \beta\sigma$$

$$A\sigma = \beta\delta + \alpha\sigma$$

Is it possible to arrange it so that $\{v, \delta, \sigma\}$ are orthonormal? Call $O = [v \quad \delta \quad \sigma]$, what is $O^T A O$

Problem 5

We have

$$A(\delta + i\sigma) = (\alpha + i\beta)(\delta + i\sigma)$$

Expand and compare real and imaginary parts to get

$$A\delta = \alpha\delta - \beta\sigma$$

$$A\sigma = \beta\delta + \alpha\sigma$$

Problem 5

By spectral theorem we know that,

$$\langle (\delta + i\sigma)\, v \rangle = 0$$

We conclude that

$$\langle \delta\, v \rangle = \langle \sigma\, v \rangle = 0$$

Now let

$$O = [v \quad \delta \quad \sigma]$$

Problem 5

We can observe

$$AO = [Av \quad A\delta \quad A\sigma]$$

$$AO = [v \quad \delta \quad \sigma] \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}$$

Thus

$$O^T AO = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{bmatrix}$$

MA106 Linear Algebra 2022

Tutorial 6 D1-T4

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Problem 1

A matrix is said to be nilpotent if $A^k = 0$ for some $k \in \mathcal{N}$.

- Show that is A is nilpotent, I-A is invertible
- What can you say about the eigenvalues of A? What is the characteristic equation of a nilpotent matrix?
- Given matrices are all nilpotent. Find geometric multiplicities of eigenvalues in each case

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Product of commuting nilpotent matrices is nilpotent. Also show that the result fails if they do not commute.

Problem 1(i)

We need to find a clever guess for the inverse. Consider

$$B = \left(I + A + A^2 \cdots A^{k-1}\right)$$

It can be seen that $(I - A)B = I$ and thus, $A - I$ is invertible.

Problem 1(ii)

$$Ax = \lambda x \implies A^k x = 0 = \lambda^k x \implies \lambda = 0$$

All eigenvalues are 0

Problem 1(iii)

The second matrix is not nilpotent, the other 2 have geometric multiplicities 1 and 2 respectively.

Problem 1(iv)

Consider

$$A^m = 0, B^n = 0, AB = BA$$

WLOG, $n < m$,
Thus

$$(AB)^n = 0$$

and thus, nilpotent
Evident counter example when $n > m$ and matrices do not commute

Problem 2

A matrix is said to be idempotent or a projection if $P^2 = P$

- Show that if P has this property so does I-P
- What can you say about the eigenvalues of P?
- If P is invertible then P=I
- Suppose P is non invertible and $v_1, v_2..v_k$ is a bsis for null space of P. Complete it to a basis $v_1, v_2..v_k, v_{k+1}, ...v_n$. Prove or disprove $Pv_k, Pv_{k+1}, ...Pv_n$ are linearly independent. Can you deduce from this that P is diagonalisable

Problem 2(i)

$$(I - P)^2 = I - 2P + P^2 = I - 2P + P = I - P$$

Problem 2(ii)

$$Px = \lambda x$$

$$P^2 x = Px\lambda x$$

$$\lambda = \lambda^2 \implies \lambda = 0, 1$$

Problem 2(iii)

$$P^{-1}P = I$$

$$P^2 = P$$

$$P^{-1}P^2 = I$$

$$P^{-1}P \times P = I$$

$$P = I$$

Problem 2(iii)

$$P^{-1}P = I$$

$$P^2 = P$$

$$P^{-1}P^2 = I$$

$$P^{-1}P \times P = I$$

$$P = I$$

Problem 2(iv)

Suppose

$$c_{k+1}Pv_{k+!}...c_nPv_n = 0$$

$$P(c_{k+1}v_{k+!}...c_nv_n) = 0$$

So $c_{k+1}v_{k+!}...c_nv_n$ lies in null space of P

$$c_{k+1}v_{k+!}...c_nv_n = a_1v_1 + ..a_nv_n$$

But basis vectors are independent!
Contradiction

Problem 3

Consider the matrix of relection and that of projection, as discussed in Tutorial 0

$$H = I - 2nn^T, H_0 = I - nn^T$$

Find eigenvalues and eigenvectors of H and H_0. Are they diagonalizable? Give reasons. Is H_0 idempotent? Try this out in 2 different ways.

Problem 3

For H:
We obtain eigenvalue -1 with eigenvector n and eigenvalue $+1$ with eigenspace as the vector subspace perpendicular to n. This makes H diagonalisable.
For H_0
We obtain eigenvalue 0 for eigenvector n and eigenvalue $+1$ with eigenspace as the vector subspace perpendicular to n. This makes H diagonalisable. Both are diagonalisable(geometric multiplicity equals algebraic for each eigenvalue) and H_0 is idempotent as $H_0^2 = H_0$

Problem 4

Are the following matrices similar? Why?

$$\begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^3 & 10^{10} & 1 \end{bmatrix}$$

Problem 4

Same eigenvalues and both are diagonalisable.
Diagonalising both gives $diag(2, 1, 3)$ and $diag(3, 2, 1)$ respectively
Thus,

$$\exists T \ s.t. \ T^{-1}AT = B$$

Problem 4

Now suppose its given that determinant of A is one.
We know that the product of all eigenvalues is nothing but the determinant of the matrix. Since our characteristic polynomial is a three dgree one with real coefficients we get two possible cases
Case 1 : All roots real
Suppose none of the roots are 1, Then the product would be $(-1)^3 = (-1) \neq det(A) = 1$ Contradiction
Case 2 : One real root and two complex conjugate roots
Let the real root be α and complex roots be $\beta, \bar{\beta}$, we have

$$\alpha\beta\bar{\beta} = 1$$

$$\implies \alpha = 1$$

Hence Proved

Problem 5

What can you say about the eigenvalues of a skey symmetric matrix? Is a skew-symmetric matrix diagonalisable?

Problem 5

$$MX = \lambda x$$
$$\bar{M} = M$$
$$M\bar{x} = \bar{\lambda}\bar{x}$$

Taking transpose and post multiplying by x

$$(M\bar{x})^T x = (\bar{\lambda}\bar{x})^T x$$

Simplifying the above

$$\bar{\lambda} = -\lambda$$

Problem 5

Diagonalisability depends upon the diagonalisability of the corresponding Hermitian matrix

Problem 6

Let $f : R^3 \mapsto R^3$ be a function such that $f(0) = 0$ and $||f(x) - f(y)|| = ||x - y||$. Is it true that $f(x) = Ax$ for some 3×3 matrix A? What kind of matrix is A?

Problem 6

Utilising

$$||Ax|| = x^T A^T AX$$

We can conclude that

$$A^T A = I$$

Conclusion

Thank you!