Problems for Wednesday Tutorial
March 23, 2022

Mainly Review.

(1) Let $\hat{a}, \hat{b}$ be unit vectors in $\mathbb{R}^3$
Discuss whether the equation $\hat{a} \times x = \hat{b}$ has solutions in $\mathbb{R}^3$; $\times$ is the cross product

(2) Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $p = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$
$(x_1^2 + x_2^2 = 1)$
What can you say about the set $\{p, Ap, A^2p, \dots\}$. Is it a finite set or an infinite set?

(3) Consider the equation $x^2+y^2-z^2+7xy-3yz+6xz = 3$
write it in the form $[x\ y\ z]\, A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
for some (3×3) symmetric matrix A.
Is such a matrix A unique? What if we drop the symmetry requirement??

(4) Recall the notion of an invertible matrix from Class 12.
How would ~~Can~~ you decide whether a 3×3 matrix is invertible or not. ?
If u is a unit vector in $\mathbb{R}^3$ (column vector)
Is $I - uu^T$ invertible?
Can you discuss the map $f : \mathbb{R}^3 \to \mathbb{R}^3$
$f(x) = (I - 2uu^T)x$
geometrically?

(5) Find two mutually $\perp$ unit vectors $\hat{u}, \hat{v}$ ~~s.t.~~ s.t. $\hat{u}, \hat{v}$ lie on the plane $x+y+z=0$. Write out a parametrization for circle $x^2+y^2+z^2=1$, $x+y+z=0$.

Consider the 4×7 System $Ax=b$:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 2 & -1 & 1 & -2 & -1 & 1 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \\ 1 & -1 & 1 & -2 & 0 & -5 & -4 & -3 \\ 2 & 2 & 6 & 0 & 4 & 2 & 4 & 10 \end{array}\right]$$

Perform Elem. Row Op. Do Gauss Elim.

Use REF to answer the following que:

(i) Rank $A = K$ value of $k =$? Nullity $A =$?

(2) Is the system solvable?

(3) Find a $(k \times k)$ Submatrix of A with non zero det

(4) Find a basis for $N(A)$ (Null sp. A)

(5) Find a basis for Column Space of A

(6) Find the Complete set of Solutions of $Ax=b$.

(7) Which are the Free Variables?

Note: Your Method must be general enough and applicable to say a 1000×10^5 matrix

Ex. for (3) Randomly picking some (2×2) Submatrix say $\begin{bmatrix} 4 & 2 \\ 0 & -5 \end{bmatrix}$ is not good enough. There must be a persuasive reason for your choice of the $(k \times k)$ Submatrix. You MUST USE REF Cleverly.

1) Determinants (Additional Exercises)

Suppose A, B are $(n \times n)$ real matrices

Such that $A + iB$ is invertible

Show that $\det \begin{pmatrix} A & B \\ -B & A \end{pmatrix} > 0.$

(Imp. in geometry)

2) The numbers 20604, 53227, 25755, 20927 and 78421 are all divisible by 17. Show that

$$\det \begin{bmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 7 & 8 & 4 & 2 & 1 \end{bmatrix}$$

is also divisible by 17.

3) Show that a Necessary Condition for

$x^2 + ax + b = 0$ to have a

$x^2 + px + q = 0$

Common root is that

$$\begin{vmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & p & q & 0 \\ 0 & 1 & p & q \end{vmatrix} = 0.$$

Hint: look up 4 linear equations in y_1, y_2, y_3, y_4 with non trivial solution

$c^3, c^2, c, 1$

(4) Tut Sheet 3: Q2, Q3, Q9, Q11.

Problems for April 13

1) Let A be a $2 \times n$ matrix of real numbers

Is there a relation between

$\det(AA^T)$ (which is a Grammian)

and the sum of the (2×2) principal minors

of A^TA

<u>Note:</u> Given a square Matrix $[P_{ij}]$ the sum of the 2×2 principal minors are

$$\begin{vmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{vmatrix} + \begin{vmatrix} P_{11} & P_{13} \\ P_{31} & P_{33} \end{vmatrix} + \dots + \begin{vmatrix} P_{n-1,n-1} & P_{n-1,n} \\ P_{n,n-1} & P_{nn} \end{vmatrix}$$

Is there a similar result when A is $3 \times n$ matrix?

Any Guess?

2) Show that

$$\begin{vmatrix} \cos\alpha & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 2\cos\alpha & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 2\cos\alpha & 1 & 0 & \cdots & 0 \\ & \cdots & & \cdots & \cdots & & 1 \\ 0 & 0 & \cdots & & & 1 & 2\cos\alpha \end{vmatrix} = \cos n\alpha.$$

3) Suppose $\langle , \rangle$ is a hermitian product

$$4\langle x, y\rangle = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$$

4) Do Q8 in Sheet 4
Q4

5) Show that if A is a $(n\times n)$ complex matrix whose rows are orthonormal then so are its cols.

6) $V = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ i \\ -1 \end{bmatrix}$; $W = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ -2i \\ -1 \end{bmatrix}$. Find $u \in \mathbb{C}^3$ s.t. $\{v, w, u\}$ orthonormal

Any way other than Gram-Schmidt?

Problems for Wednesday April 20

(1) Suppose P is non singular $(n \times n)$, A, B are both $(n \times n)$ matrices. Show that A and $P^{-1}AP = B$ have same Char. Eq^n.

Tell them the terminology: Similar Matrices

(2) Show that if A, B are square matrices of the same size $(n \times n)$

$I_n - AB$ invertible iff $I_n - BA$ invertible

Do AB and BA have the same eigen values?

(3) Prove that if Nullity of A is k then x^k divides Ch. poly $\det(xI - A)$

Hint: Determinantal Rank. How do you find. Coeff. of Char. Polynomial?

~~Deduce that~~ Nullity $(A - \lambda I) = k$ $\Rightarrow$ λ is an eigen value that

(~~and~~) has multiplicity at least k as a root of the Char. polynomial

(4) Find Eigen Values and Eigen vectors of

$$\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} ; \quad \begin{bmatrix} 4 & -1 & -2 \\ 2 & 1 & -2 \\ 1 & -1 & 1 \end{bmatrix}$$

(5) Go over the quiz problems since this Tutorial will take place AFTER the quiz.

Problems for April 27

I) Identify the Quadrics and in atleast one instance find the directions of the principal axes

(i) $2xy + 2yz + 2zx = 1$

(ii) $x^2 - 2y^2 + 4z^2 + 6yz = 1$

(iii) $-x^2 - y^2 + 2z^2 + 8xy - 4xz + 4yz = 1$

II) Compute $\int \exp -(2x^2 + 5y^2 + 2z^2 - 4xy - 2xz + 4yz)\, dx\,dy\,dz$

III) Show that $ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz$ factorizes into a product of linear factors (possibly with Complex Coeff)

iff $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$

Hint. First discuss the case when $f = g = h = 0$

(IV) Show that a (3×3) orthogonal matrix has eigen value $+1$ or -1

Further, if det of Matrix is 1 then 1 is necessarily an eigen value.

(V) Let A be a 3×3 Orthogonal real matrix and v be a unit vector with eigen value ± 1 or -1 }.

Let $\alpha + i\beta$ be a Complex eigen value and $\rho + i\sigma$ be the Corresp. eigen vector. in $\mathbb{C}^3$

prove that $A\rho = \alpha\rho - \beta\sigma$

$A\sigma = \beta\rho + \alpha\sigma$

Is it possible to arrange it so that $\{v, \rho, \sigma\}$ orth. normal?

Call $O = [v\ \rho\ \sigma]$ What is $O^T A O$?

Tut problems May 4, 2022

(1) A matrix A is said to be nilpotent if $A^k = 0$ for some $k \in \mathbb{N}$

(a) Show that if A is nilpotent, $I - A$ is invertible

(b) What can you say about eigen values of A? What is the Char. Eqn of a nilpotent matrix?

(c) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

are all nil ptt. Find geom. Mult of its eigen value in each case

(d) product of ~~nil~~ Commuting nil ptt matrices is nilpotent. Show that the result fails if the matrices do not Commute.

(2) A matrix P is said to be an idempotent or a projection if $P^2 = P$.

a) Show that P has this property so does $I - P$

b) What can you say about eigen values of P?

c) If P is invertible then $P = I$

d) Suppose P is not invertible and $v_1, \ldots, v_k$ is a basis for Null Sp. P.

Complete it to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$

Prove or disprove $Pv_{k+1}, \ldots, Pv_n$ Lin Indep.

Can you deduce from this that P is diagonalizable?

(3) Consider the matrix of Reflection about a plane encountered in Tut Sheet 1

$$H = I - 2nn^T \;;\; (\|n\| = 1)$$

and

$$H_0 = I - nn^T$$ (both H, H_0 were discussed)

Find Eigen values of H and H_0.

Are they diagonalizable? Give Reasons.

Is H_0 an idempotent? Try this out in two ways ~~Could you have~~ using geometric reasoning and verifying algebraically

(4) Are the matrices $\begin{bmatrix} 2 & 10^5 & 10^9 \\ 0 & 1 & \pi \\ 0 & 0 & 3 \end{bmatrix}$ and $\begin{bmatrix} 3 & 0 & 0 \\ e & 2 & 0 \\ 10^8 & 10^{10} & 1 \end{bmatrix}$ similar? Why?

(5) What can you say about the eigen values of a skew symmetric matrix?

Is a skew symm matrix diagonalizable.

(6) Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be a function s.t. $f(0) = 0$ and $\|f(x) - f(y)\| = \|x - y\|$ ~~$\|f(x)\| = \|x\|$~~

Is it true that $f(x) = Ax$ for some 3×3 matrix A? What kind of matrix is A?