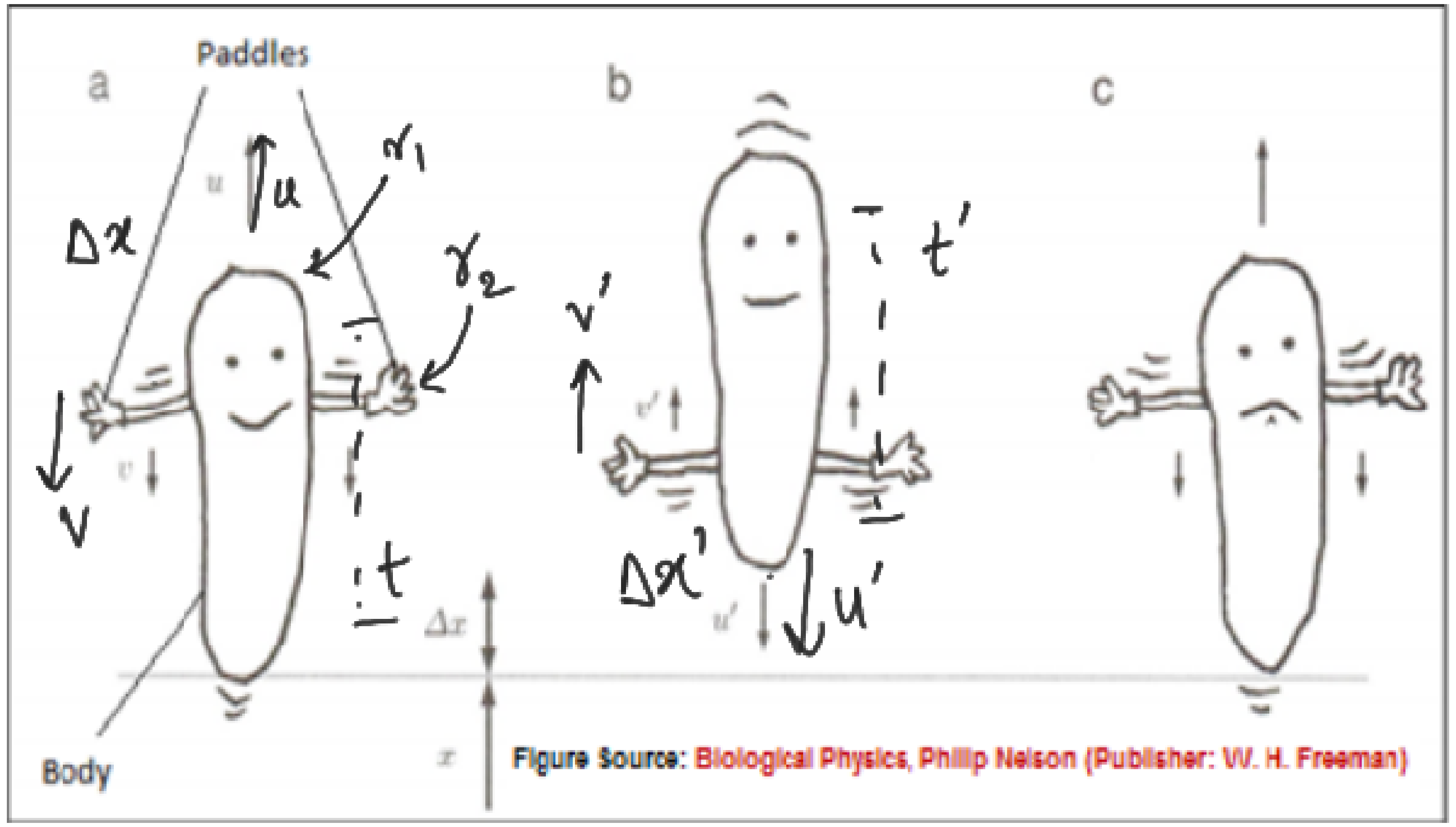
1. Consider a microscopic swimmer trying to make progress by cycling between the upward and downward strokes of its paddles as shown below. (a) On the first stroke, the paddle move downward at relative speed v, propelling the body through the fluid upward at speed u. (b) On the second stroke, the paddle move upward at relative speed v', propelling the body downward at speed u' (c) Then the cycle repeats. Assume this is low Reynolds number motion where moving the body through the fluid requires a force determined by drag coefficient γ_1 and moving the paddles through the fluid requires a force determined by a different constant γ_2. Show that reciprocal motion like this cannot give net progress in low Reynolds number environment.



First stroke

$\vec{V}_{paddle} = u\hat{i} - v\hat{i} = (u-v)\hat{i}$

$(-\gamma_1 u\hat{i}) - \gamma_2(u-v)\hat{i} = 0$

$\Rightarrow -(\gamma_1+\gamma_2)u\hat{i} + \gamma_2 v\hat{i} = 0$

$\Rightarrow u = \dfrac{\gamma_2 v}{\gamma_1+\gamma_2}$ — (1)

Second stroke

$\vec{V}_{paddle} = -u'\hat{i} + v'\hat{i} = (v'-u')\hat{i}$

$(\gamma_1 u'\hat{i}) - \gamma_2(v'-u')\hat{i} = 0$

$\Rightarrow (\gamma_1+\gamma_2)u'\hat{i} - \gamma_2 v'\hat{i} = 0$

$\Rightarrow u' = \dfrac{\gamma_2 v'}{\gamma_1+\gamma_2}$ — (2)

$\vec{V}_{paddle} = \vec{V}_{body} + \vec{V}_{relative}$

$\vec{f}_{body} + \vec{f}_{paddle} = 0$

(microscopic body)

Force balance for whole system

Geometrically reciprocal => Distance covered by paddle is same => $\underline{tv = t'v'}$ ⟶ distances of paddle

③ $u = \frac{r_1 v}{r_1 + r_2}$, $u' = \frac{r_1 v'}{r_1 + r_2}$

To show => $\Delta x = \Delta x'$

$\left.\begin{array}{l} \Delta x = tu \\ \Delta x' = t'u' \end{array}\right\}$ Distances of body

Alternate : use ① & ② and show $\frac{u}{v} = \frac{u'}{v'}$ and use this

$\Delta x' = t'u' = (tv/v')u'$

$= tv \times \frac{u'}{v'} = tv \times \frac{r_1}{r_1 + r_2}$

$= tu = \underline{\underline{\Delta x}}$

2. Solution of one dimensional diffusion equation for a substance freely diffusing with initial condition $C(x,0) = C_0\delta(x - x_0)$ is given by $C(x,t) = \frac{C_0}{\sqrt{4\pi Dt}} e^{-\frac{(x-x_0)^2}{4Dt}}$. Write down the solution of diffusion equation for a substance diffusing in presence of a perfectly reflecting wall located at $x = 0$ i.e. $\vec{J}(0,t) = 0$. Draw the resulting concentration profile in presence of perfectly reflecting wall located at $x = 0$ for $t > 0$. (Hint: Consider an imaginary source located at $-x_0$ and make required adjustment to real source).

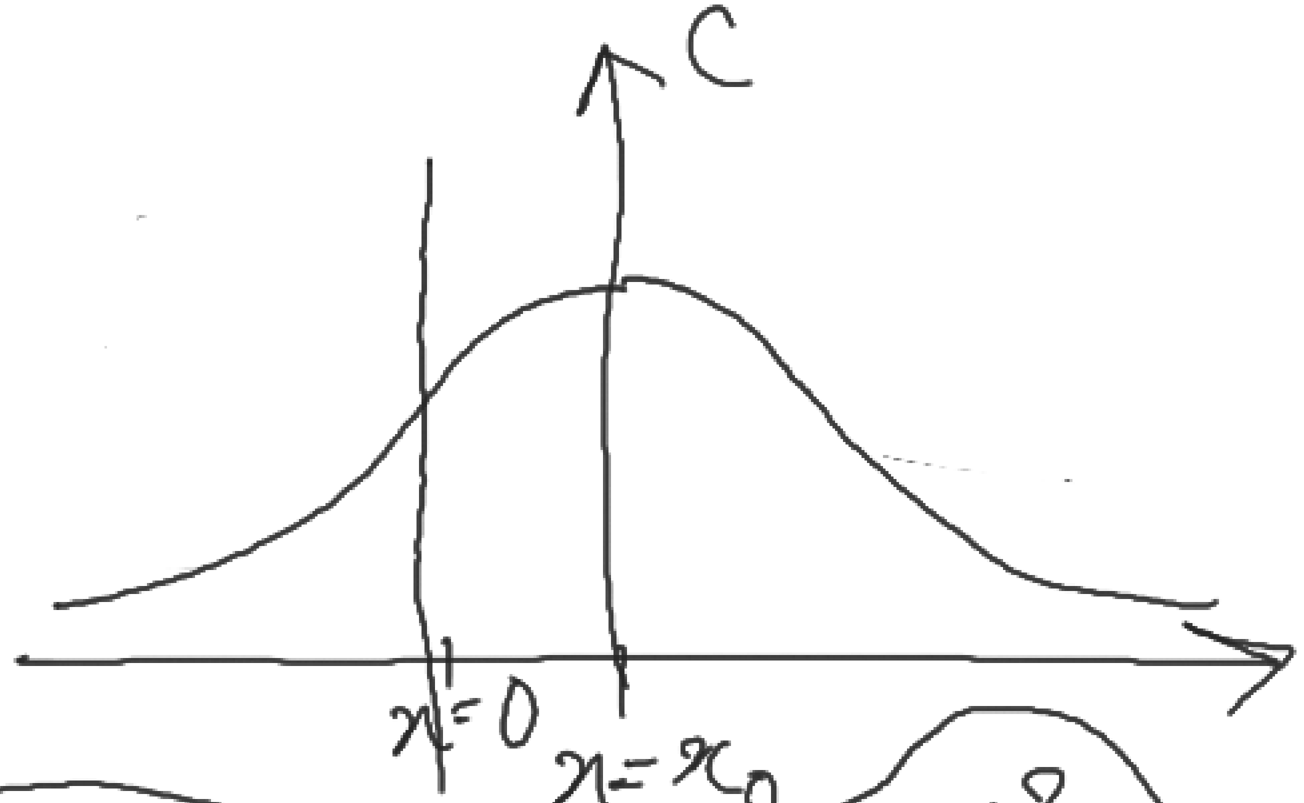
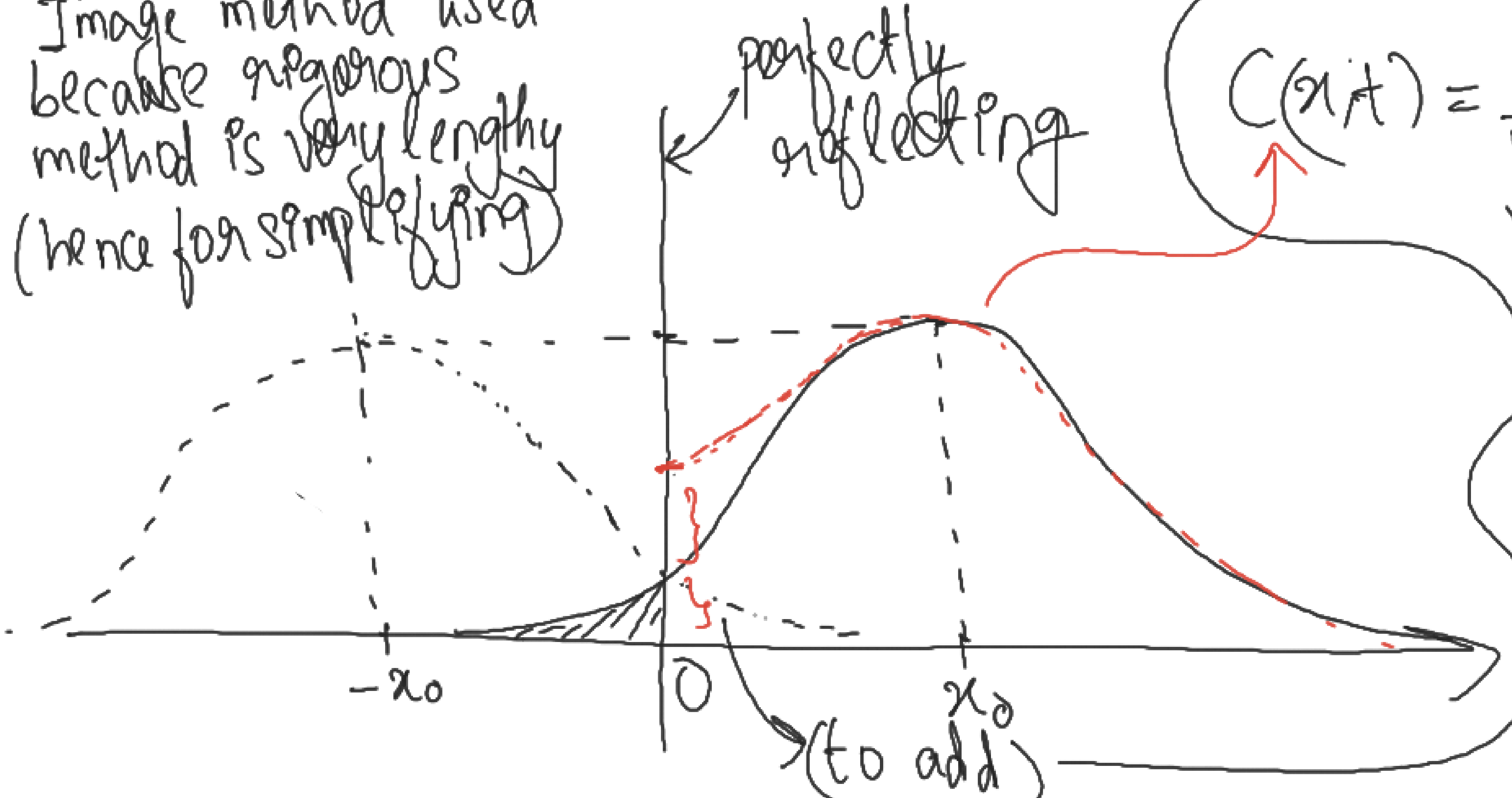


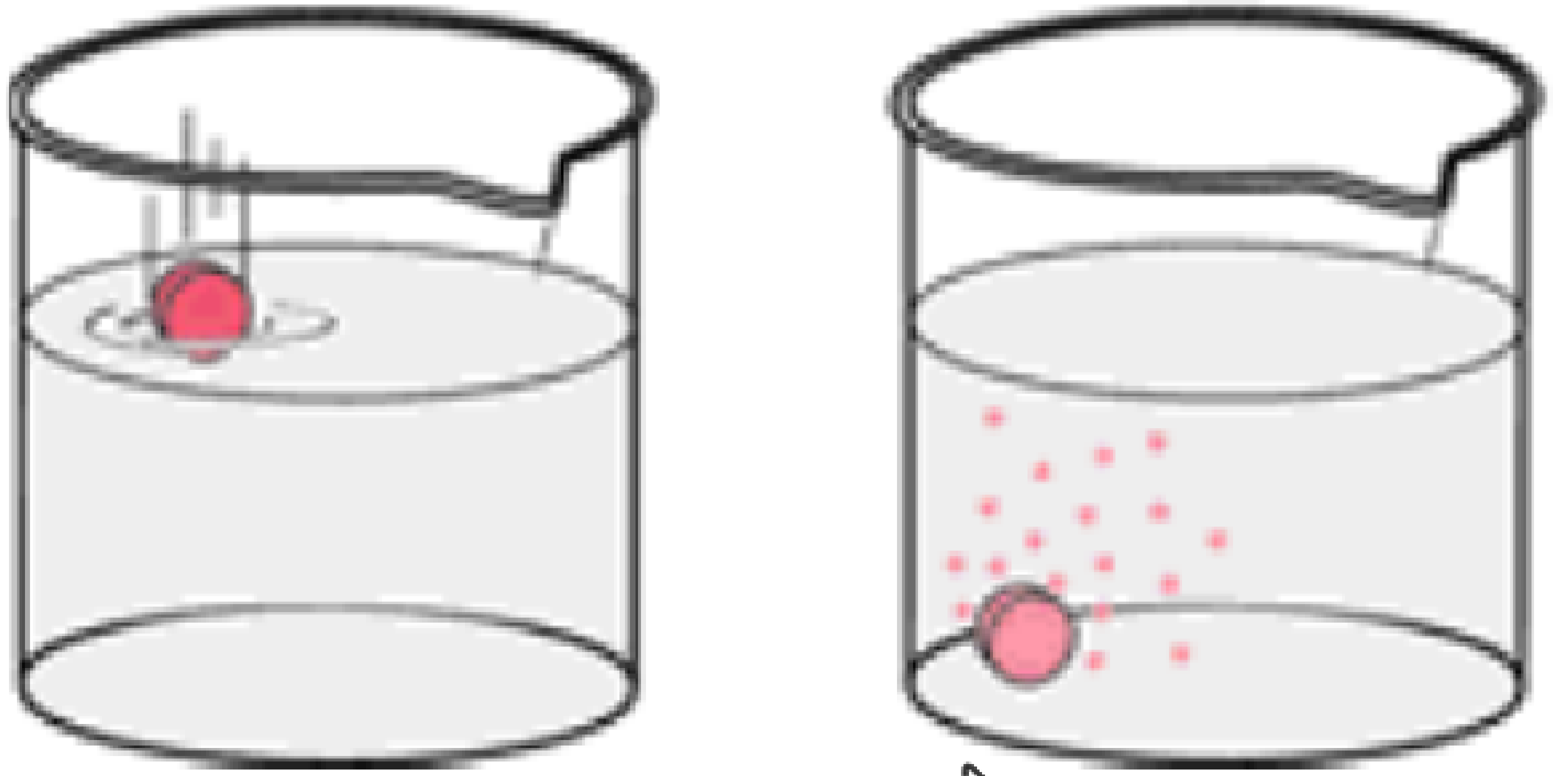
Image method used because rigorous method is very lengthy (hence for simplifying)

$$C(x,t) = \frac{C_0}{\sqrt{4\pi Dt}} e^{-\frac{(x-x_0)^2}{4Dt}} + \frac{C_0}{\sqrt{4\pi Dt}} e^{-\frac{(x+x_0)^2}{4Dt}}$$

$(x \geq 0)$



3. Suppose that drug molecules diffuse out of a tablet (which is modelled as a thin plane wall) into a solution. In addition to diffusion, the drug molecules also undergo a chemical reaction which causes drug to deplete with time in proportion to its present concentration. The constant rate of the chemical reaction which depletes the drug molecules with time is k and D is the diffusion constant of the drug. Find out the concentration profile for the drug as a function of distance x away from the tablet wall in the solution in the steady state i.e. when concentration doesn't change with time. Show that drug will be drawn out of the tablet rapidly if it has high diffusion constant or has a high reaction rate in solution. (*Hint: Add one extra term to diffusion equation to include the effect of depletion of drug due to chemical reaction*)



Assume concentration of tablet at the wall (at $x=0$) is C_0

Diffusion $\Rightarrow \frac{\partial C}{\partial t} \propto \frac{\partial^2 C}{\partial x^2}$ $\Rightarrow \frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - kC = 0$ (steady state)

Depletion $\Rightarrow \left|\frac{\partial C}{\partial t}\right| \propto C$ $\Rightarrow \frac{\partial^2 C}{\partial x^2} = \frac{k}{D} C \Rightarrow$ General solution

$$C = A_1 e^{-B_1 x} + A_2 e^{+B_2 x}$$

Constants $\Rightarrow$ D and k Boundary conditions: $\underline{C(x=0) = C_0}$, $\underline{C(x\to\infty) \to 0}$

$C(x \to \infty) = 0 \Rightarrow A_2 = 0 \Rightarrow C(x) = A_1 e^{-B_1 x}$

$C(x=0) = C_0 \Rightarrow A_1 = C_0 \Rightarrow \underline{C(x) = C_0 e^{-B_1 x}}$

Finding B_1:

$$\frac{\partial^2 C}{\partial x^2} = \frac{K}{D} C$$

$$\Rightarrow B_1^2 = \frac{K}{D} \Rightarrow \underline{B_1 = \sqrt{\frac{K}{D}}}$$

$$\boxed{C = C_0 e^{-\sqrt{\frac{K}{D}} x}} \rightarrow \underline{\text{conc. profile}}$$

Rate of drawing out = Flux$(x=0)$

$$\Rightarrow J = -D \frac{\partial C}{\partial x}\Big|_{x=0} = -D\left(-\sqrt{\frac{K}{D}}\, C_0 e^{-\sqrt{\frac{K}{D}} x}\right)_{x=0} = \boxed{\sqrt{KD}\, C_0}$$

$\downarrow$

Physical significance: larger $K \Rightarrow$ larger gradient due to faster reduction of conc.

larger $D \Rightarrow$ more diffusion $\Rightarrow$ more flux